PLAN

Introduction

- Introduce myself
- Format: short talk + go through questions
  → interrupt me at any point!

Questions?

RF: rotations & ref. frames

Go through the questions!

ACP Rapid Feedback 1 : Part I

There is very rich and interesting geometry and other maths behind tensors. We will think of them as multi-linear maps from vector spaces, so given a coordinate system we may represent them as multidimensional matrices. The transformation properties are crucial, so not any matrix is a tensor, e.g.

$$\overrightarrow{\text{fruits}} = \begin{pmatrix} \# \text{ of apples} \\ \# \text{ of oranges} \end{pmatrix}$$

is not a spatial vector, even though it looks like one. Given any vector, it's a matter of my choice of coordinates what the components of the vector will actually be!

For a (p,q)-tensor T, we use index notation to denote its $(i_1, \dots, i_p, j_1, \dots, j_q)$-th element as

$$T^{i_1 \dots i_p}{}_{j_1 \dots j_q}$$

be aware that e.g.

$$T^{\alpha\beta} \neq T^{\beta\alpha}$$

$$T_{\alpha\beta} \, U_{\mu\nu} = U_{\mu\nu} \, T_{\alpha\beta}$$

It's generally important to distinguish up from down indices, and we use Einstein summation convention when they're paired up, i.e.

$$S^{\alpha\beta} \, T_{\alpha\gamma} = \sum_{\alpha=1}^{D} S^{\alpha\beta} \, T_{\alpha\gamma}$$

However on flat Cartesian space we use the metric with which we raise / lower indices is $g = \delta$, so at a component level

$$v^i = v_i$$

ACP Problem Sheet 1 : Part I

1. Consider the $n \times n$ matrices A, B, C, D. Let $i, j, \ldots \in \{1, \ldots, n\}$. Then

$$A^i{}_j = [B(C+D)]^i{}_j = B^i{}_k (C+D)^k{}_j = B^i{}_k (C^k{}_j + D^k{}_j)$$

$$\bar{A}^i{}_j = [BCD]^i{}_j = B^i{}_m C^m{}_n D^n{}_j$$

2.

a) - h)

$$A_i B^i = \overset{a)}{\delta^{ij} A_i B_j} = \overset{c)}{A_i \delta^{ij} B_j} = \overset{b)}{A_i B_j \delta^{ij}}$$

$$= \overset{e)}{\delta^{ji} A_i B_j} = \overset{d)}{A_i \delta^{ji} B_j} = A_i B_j \delta^{ji}$$

$\delta^{ij} = \delta^{ji}$

dummy variable

$$= \overset{f)}{\delta^{kl} A_k B_l}$$

$$= \overset{g)}{A^i B_i} = \overset{h)}{B_i A^i}$$

i) $A_{ijk} B^{jk} = A_{ikj} B^{kj}$

j) $u_i v_j w_k = u_i w_k v_j$

k) $u \cdot v = v \cdot u$

l) $u^i = A^i{}_j v^j = A^{ij} v_j$

m) $A_{ij} v^j = v^j A_{ij} \neq v^i A_{jj}$

n) $A_{ij} v^j = A_{ik} v^k = v^k A_{ik} \neq v^k A_{ki}$

o) - q)

$$A_{ij} B^{ij} = \underset{o)}{A_{ji} B^{ji}} = \underset{q)}{A^i{}_j B_i{}^j} \neq \underset{q?}{A^i{}_j B^j{}_i}$$

unless $A_{ij} = A_{ji}$ or $B_{ij} = B_{ji}$

r? $A_{ij} B^{ij} \neq A^i{}_j B^i{}_j$

3. consider the 2x2 matrix

skip

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Then

$$\lambda_1 = (\mathrm{Tr}\, A)^2 = (A^i{}_i)^2 = \left(\sum_{i=1}^{2} A^i{}_i\right)^2$$

$$= (A^1{}_1 + A^2{}_2)^2 = (A_{11} + A_{22})^2 = \underline{\underline{(a+d)^2}}$$

$$\lambda_2 = \mathrm{Tr}(A^2) = \mathrm{Tr}(A^i{}_j A^j{}_k) = A^i{}_j A^j{}_i$$

$$= \sum_{i,j=1}^{2} A^i{}_j A^j{}_i$$

$$= A^1{}_1 A^1{}_1 + A^1{}_2 A^2{}_1 + A^2{}_1 A^1{}_2 + A^2{}_2 A^2{}_2$$

$$= A_{11}^2 + A_{12} A_{21} + A_{21} A_{12} + A_{22}^2$$

$$= \underline{\underline{a^2 + 2bc + d^2}}$$

$$\lambda_3 = A_{ij} A^{ij} = \sum_{i,j=1}^{2} A_{ij} A^{ij}$$

$$= A_{11} A^{11} + A_{12} A^{21} + A_{21} A^{21} + A_{22} A^{22}$$

$$= A_{11}^2 + A_{11} A_{12} + A_{21} A_{21} + A_{22} A_{22}$$

$$= \underline{\underline{a^2 + b^2 + c^2 + d^2}}$$

Can e.g. use this to verify that

$$(\mathrm{Tr}\, A)^2 - \mathrm{Tr}\,(A^2) = (\lambda_1 + \lambda_2)^2 - (\lambda_1^2 + \lambda_2^2)$$

$$= 2\lambda_1 \lambda_2 = 2 \det A$$

4.

$$[\underline{r} \times (\underline{\omega} \times \underline{r})]_i = \varepsilon_{ijk}\, r^j\, (\underline{\omega} \times \underline{r})^k$$

$$= \varepsilon_{ijk}\, r^j\, \varepsilon^k{}_{lm}\, \omega^l r^m$$

$$= \varepsilon_{ijk}\, \varepsilon_{lm}{}^k\, \omega^l r^m\, r^j$$

$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\, \omega^l r^m\, r^j$$

$$= \omega_i\, \delta_{jm} r^j r^m - \omega_j\, r^j\, r_i$$

$$= \underline{(|\underline{r}|^2 \delta_{ij} - r_i r_j)}\, \omega_j$$

ACP Rapid Feedback: Part II

For the position

$$\frac{d\underline{r}}{dt} = \underline{\omega} \times \underline{r}$$

Let $\underline{A}$ have explicit time-dependence, and let $\bar{e}_i$ be the basis of the rotating frame

$$\left.\frac{d\underline{A}}{dt}\right|_I = \frac{d}{dt}\left(A^i \bar{e}_i \right)$$

$$= \frac{dA^i}{dt}\bar{e}_i + A^i \frac{d\bar{e}_i}{dt}$$

$$= \frac{dA^i}{dt}\bar{e}_i + A^i \left(\underline{\omega} \times \bar{e}_i\right)$$

$$= \left.\frac{d\underline{A}}{dt}\right|_R + \underline{\omega} \times \underline{A}$$

Similarly,

$$\left.\frac{d^2\underline{r}}{dt^2}\right|_I = \left.\frac{d}{dt}\right|_I \left.\frac{d\underline{r}}{dt}\right|_I$$

$$= \left.\frac{d}{dt}\right|_I \left(\left.\frac{d\underline{r}}{dt}\right|_R + \underline{\omega} \times \underline{r} \right)$$

$$= \left.\frac{d\underline{v}_R}{dt}\right|_I + \left.\frac{d\underline{\omega}}{dt}\right|_I \times \underline{r} + \underline{\omega} \times \left.\frac{d\underline{r}}{dt}\right|_I$$

$$= \left.\frac{d\underline{v}_R}{dt}\right|_R + \underline{\omega} \times \underline{v}_R + \dot{\underline{\omega}} \times \underline{r} + \underline{\omega} \times \left(\underline{v}_R + \underline{\omega} \times \underline{r} \right)$$

$$= \left.\frac{d^2\underline{r}}{dt^2}\right|_R + \underset{\text{Coriolis}}{2\underline{\omega} \times \left.\frac{d\underline{r}}{dt}\right|_R} + \underset{\text{Centrifugal}}{\underline{\omega} \times (\underline{\omega} \times \underline{r})} + \underset{\text{Euler}}{\dot{\underline{\omega}} \times \underline{r}}$$

ACP Problem sheet 2 : Part II

7. Consider a particle of charge q moving around a fixed charge $-q'$ in uniform mag. field $\underline{B}$.

a) For an inertial observer

$$m \left.\frac{d^2\underline{r}}{dt^2}\right|_I = -\frac{k}{r^2}\hat{\underline{r}} + q \left.\frac{d\underline{r}}{dt}\right|_I \times \underline{B}$$

For a rotating observer,

$$m \left.\frac{d^2 r}{dt^2}\right|_R = m\left[\left.\frac{d^2 r}{dt^2}\right|_I - 2\underline{\omega} \times \left.\frac{dr}{dt}\right|_R - \underline{\omega}\times(\underline{\omega}\times\underline{r}) - \dot{\underline{\omega}}\times\underline{r} \right]$$

$$= -\frac{k}{r^2}\hat{\underline{r}} + q\left(\left.\frac{d\underline{r}}{dt}\right|_R + \underline{\omega}\times\underline{r}\right)\times\underline{B}$$

$$-2m\underline{\omega}\times\left.\frac{d\underline{r}}{dt}\right|_R - m\underline{\omega}\times(\underline{\omega}\times\underline{r}) - m\dot{\underline{\omega}}\times\underline{r}$$

$$= -\frac{k}{r^2}\hat{\underline{r}} + q(\underline{v}_R + \underline{\omega}\times\underline{r})\times\underline{B} - 2m\underline{\omega}\times\underline{v}_R - m\underline{\omega}\times(\underline{\omega}\times\underline{r})$$

$$-m\dot{\underline{\omega}}\times\underline{r}$$

$$= -\frac{k}{r^2}\hat{\underline{r}} + \underline{v}_R\times\left(q\underline{B} + 2m\underline{\omega}\right) + q(\underline{\omega}\times\underline{r})\times\underline{B}$$

$$-m\underline{\omega}\times(\underline{\omega}\times\underline{r}) - m\dot{\underline{\omega}}\times\underline{r}$$

b) Rotating frame is a good choice of frame, since everything simplifies when

$$\underline{\omega} = -\frac{q}{2m}\underline{B}$$

Complicated if $\dot{\underline{B}} \neq \underline{0}$, so let's take $\dot{\underline{B}} = \underline{0}$. Then

$$m\left.\frac{d^2 r}{dt^2}\right|_R = -\frac{k}{r^2}\hat{\underline{r}} + q\cdot -\frac{q}{2m}(\underline{B}\times\underline{r})\times\underline{B} - m\cdot\left(-\frac{q}{2m}\right)^2\underline{B}\times(\underline{B}\times\underline{r})$$

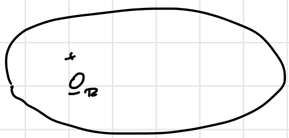
$$= -\frac{k}{r^2}\hat{\underline{r}} - \frac{q^2}{4m}\left[2(\underline{B}\times\underline{r})\times\underline{B} + \underline{B}\times(\underline{B}\times\underline{r})\right]$$

$$= -\frac{k}{r^2}\hat{r} - \underbrace{\frac{q^2}{4m}(B \times r) \times B}$$

c) Ignoring quadratic terms, when mag. fields are weak

$$m a_R = -\frac{k}{r^2}\hat{r}$$

which is the e.o.m. of a particle in a central potential in D=4. These are perfect ellipses.



d) An inertial observer sees this rotating at ω. When $|\omega|^{-1}$ is larger than the period of an ellipse, the ellipse just precesses - not necessarily in the plane of the ellipse!

This is known as Larmor precession.

Note! we found that there is a precession of the orbit described by

$$\omega = -\frac{q}{2m} B$$

Define the gyromagnetic ratio via

$$\mu = \gamma L$$

where

$$\underline{\tau} = \underline{\mu} \times \underline{B}$$

it turns out that this shows that

$$\gamma = \frac{gq}{2m}, \qquad g = 1$$

precisely. very interestingly, the quantum mechanical prediction for a Dirac fermion, e.g. electron is

$$g_e = 2$$

The QFT corrections to this are

$$g_e = 2\left(1 + \frac{\alpha}{2\pi} + \dots\right)$$

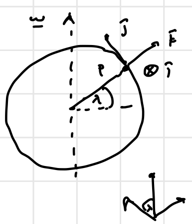
where the first subleading term was found by J. Schwinger and is engraved on his tombstone. The corrections are known up to $O(\alpha^5)$ and matches experiments up to 12 DP.

This is the most accurately verified prediction in all of science!

The anomalous magnetic dipole moment of muons is a useful measurement for new physics!

2. Foucault pendulum

a) (spherical) earth, with coordinate system at P.



The Coriolis force points towards

$$\underline{v} \times \underline{\omega} \propto \hat{y} \times (\hat{y} + \hat{z}) \propto \hat{y} \times \hat{z} = \hat{x}$$

in the $\hat{x}$ direction.

b) for small angles, the pendulum swings in the x-y plane, i.e.

$$\underline{v} = \dot{x}\,\hat{i} + \dot{y}\,\hat{j}$$

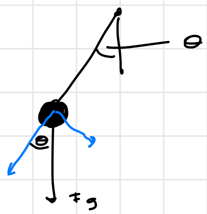
from the diagram, we see that

$$\underline{\omega} = \omega\left(\cos\lambda\,\hat{j} + \sin\lambda\,\hat{k}\right)$$

Hence

$$-2\underline{\omega}\times\underline{v} = -2\omega\left[\dot{x}\left(-\cos\lambda\,\hat{k} + \sin\lambda\,\hat{j}\right) - \dot{y}\sin\lambda\,\hat{i}\right]$$

we also have an external force due to gravity.



In the 2D plane,

$$F_{restoring} = F_g \sin\theta \approx F_g\,\theta = F_g\,\frac{\delta r}{L}$$

hence, as a vector

$$\underline{F}_{rest} = -mg\,\frac{\underline{r}}{L}$$

Together, ignoring the centrifugal and Euler force

$$\left.\frac{d^2\underline{r}}{dt^2}\right|_R = \underline{a}_R = \underline{a}_I - 2\underline{\omega}\times\underline{v}_R =$$

In components, concentrating on the x-y plane

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} -\frac{g}{L} x + 2\omega \sin\lambda \, \dot{y} \\ -\frac{g}{L} y - 2\omega \sin\lambda \, \dot{x} \end{pmatrix}$$

hence

$$\ddot{x} - 2\Omega \dot{y} + \omega_0^2 x = 0$$
$$\ddot{y} + 2\Omega \dot{x} + \omega_0^2 y = 0$$

where $\Omega = \omega \sin\lambda$ and $\omega_0^2 = g/L$. Taking $z = x + iy$, then

$$\ddot{z} = \ddot{x} + i\ddot{y} = 2\Omega(\dot{y} - i\dot{x}) - \omega_0^2 (x + iy)$$

$$= -2i\Omega \dot{z} - \omega_0^2 z$$

and

$$\underbrace{\ddot{z} + 2i\Omega \dot{z} + \omega_0^2 z = 0}$$

as required.

c) Trying

$$z = z_0 e^{i\lambda t}$$

gives

$$\left(-\lambda^2 + 2i\Omega \cdot i\lambda + \omega_0^2\right) z_0 e^{i\lambda t} = 0$$

which is solved by

$$\lambda = \frac{+2\Omega \pm \sqrt{(-2\Omega)^2 - 4 \cdot -\omega_0^2}}{2 \cdot -1} = -\left(\Omega \pm \sqrt{\Omega^2 + \omega_0^2}\right)$$

so the general solution is

$$z(t) = e^{-i\Omega t}\left(A e^{i\omega_1 t} + B e^{-i\omega_1 t}\right)$$

when

$$\omega_1 = \sqrt{\Omega^2 + \omega_0^2}$$

we start the pendulum from rest at $x = a$ and $y = 0$, so

$$z(t=0) = A + B = a$$

$$\dot{z}(t=0) = -i\Omega(A+B) + i\omega_1(A-B) = 0$$

then

$$A - B = \frac{\Omega}{\omega_1} a$$

so

$$A = \frac{a}{2}\left(1 + \frac{\Omega}{\omega_1}\right) \qquad B = \frac{a}{2}\left(1 - \frac{\Omega}{\omega_1}\right)$$

Earth rotates much more slowly than the pendulum swings, so $\Omega \ll \omega_0$, so $\Omega \ll \omega_1$, and

$$A \approx B \approx \frac{a}{2}$$

then

$$z = e^{-i\Omega t} \cdot \frac{a}{2}\left(e^{i\omega_1 t} + e^{-i\omega_1 t}\right)$$

$$= \underline{a e^{-i\Omega t} \cos(\omega_1 t)}$$

or

$$x(t) = a\cos(\Omega t)\cos(\omega_1 t)$$
$$\underline{y(t) = -a\sin(\Omega t)\cos(\omega_1 t)}$$

d) Note that in polar coordinates $(x, y) \to (\rho, \phi)$

$$\rho = \left(a^2 \cos^2(\omega_n t) \left(\cos^2 \Omega t + \sin^2 \Omega t \right) \right)^{1/2} = a \cos(\omega_n t)$$

$$\phi = \arctan\left(- \tan \Omega t \right) = - \Omega t$$

so this describes a pendulum swinging at ω_n in a plane rotating at Ω.

At $\lambda = 51.5°$, the plane rotates with period

$$T = \frac{2\pi}{\Omega} = \frac{2\pi}{\omega \sin\lambda} = \frac{1}{f \sin\lambda} \approx \frac{24 \text{ hours}}{\sin(51.5°)} = \underline{30.67 \text{ hours}}$$

This is the latitude for e.g. the Foucault pendulum in the science museum!