PLAN

Chit chat?

Who's still enjoying ACT?

Demo: Who's 3rd / 4th year UG, MSc, QFFF?

Questions

Common mistakes

Not using integrals on the sheet

Doing the integrals wrong! E.g. thinking z doesn't vary over the cone, or shifting variables / coordinates inconsistently.

Conflating rotations about different points;

Rf: inertia

Go through questions

ACP Rapid Feedback 2

We can treat many-body systems by treating the constituents individually with appropriate weighting / normalisation. We will concentrate on discrete systems, but it's easy to generalise to continuous systems.

Let $a, b, \ldots \in \{1, \ldots, N\}$ label the particles, and denote quantities for individual particles with lowercase letters. The centre of mass position is

$$\underline{R} = \sum_a m_a \underline{r}_a$$

Its derivatives (i.e. $\underline{P}$ and $\dot{\underline{P}}$) follow the known laws like point particles (as long as internal forces are central). The angular momentum is defined as

$$\underline{L} = \sum_a m_a \underline{r}_a \times \underline{\dot{r}}_a = I \, \underline{\omega}$$

which defines the inertia tensor

$$I_{ij} = \sum_a m_a \left(\underline{r}_a^2 \, \delta_{ij} - r_{a,i} \, r_{a,j} \right)$$

The kinetic energy of this is also analogous to the linear case i.e.

$$T = \frac{1}{2} \sum_a m_a |\underline{\omega}_a \times \underline{r}_a|^2 = \frac{1}{2} \sum_a m_a \underline{\omega} \cdot \left(\underline{r}_a \times (\underline{\omega} \times \underline{r}_a) \right) = \frac{1}{2} I_{ij} \omega^i \omega^j$$

Symmetry axes of I correspond to eigenvectors of I, so I is diagonal in suitably chosen coordinates.

The time-dependence of $\underline{L}$ is

$$\dot{\underline{L}} = \sum_a \underline{r}_a \times \left(\underline{f}^{ext}_a + \sum_b \underline{f}^{int}_{ab} \right)$$

$$= \sum_a \underline{r}_a \times \underline{f}_a^{ext} + \frac{1}{2} \sum_{ab} (\underline{r}_a - \underline{r}_b) \times \underline{f}_{ab}^{int}$$

For central forces, the second term drops out, and

$$\dot{\underline{L}} = \sum_a \underline{r}_a \times \underline{f}_a^{ext} = \underline{\tau}$$

we can also seperate the centre of mass motion from the rest. If we take

$$\underline{r}_a = \underline{R} + \underline{r}_a^*$$

then

$$\underline{L} = \sum_a m_a \left(\underline{R} \times \dot{\underline{R}} + \cancel{\underline{R} \times \dot{\underline{r}}_a^*} + \cancel{\underline{r}_a^* \times \dot{\underline{R}}} + \underline{r}_a^* \times \dot{\underline{r}}_a^* \right)$$

$$= M\underline{R} \times \dot{\underline{R}} + \underline{L}^*$$

$$\dot{\underline{L}}^* = \sum_i \underline{r}_i^* \times \underline{f}_i^{ext}$$

we can also do this for the moment of inertia:

$$I_{ij} = \sum_a m_a \Big[\left(|\underline{R}|^2 + \cancel{2 \underline{R} \cdot \underline{r}_a^*} + |\underline{r}_a^*|^2 \right) \delta_{ij}$$

$$- \left(R_i R_j + \cancel{2 R_{(i} r_{a\,j)}^*} + r_{a\,i}^* r_{b\,j}^* \right) \Big]$$

$$= I_{ij}^{com} + I_{ij}^*$$

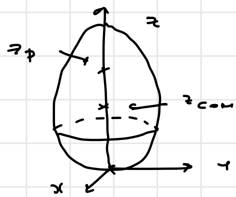
This generalises the parallel axis theorem. The kinetic energy splits too:

$$T = \frac{1}{2} \sum_a m_a \left(|\dot{\underline{R}}|^2 + 2 \dot{\underline{R}} \cdot \dot{\underline{r}}_a^* + |\dot{\underline{r}}_a^*|^2 \right)$$

$$= \frac{1}{2} M |\dot{\underline{R}}|^2 + \frac{1}{2} \sum_a m_a |\dot{\underline{r}}_a^*|^2$$

ACP Problem Sheet 2

1. Hard-boiled egg of mass M and constant density $\bar{\rho}$.



a) We know cylindrical symmetry about the z-axis, hence the z-axis is a symmetry axis (or the x-y plane is a symmetry plane). The principal axes, i.e. eigenvectors of the inertia tensor include the z-axis, and (any linear combination of) the x- and y-axis, the latter two being degenerate. Since the coordinate axes align with the principal axes, we expect I to be diagonal, and by symmetry $I_{xx} = I_{yy}$.

In particular, recall that

$$I_{ij} = \int_{\text{body}} d^3x \; \rho(r) \left(|r|^2 \delta_{ij} - r_i r_j \right)$$

For our system, the off-diagonal components ($i \neq j$) are

$$I_{ij} = -\bar{\rho} \int_{\text{egg}} d^3x \; r_i r_j$$

Since $i \neq j$, one of $r_i, r_j \in \{x, y\}$. The body and hence integration domain is even under $x \mapsto -x$, $y \mapsto -y$ by rotational symmetry while the integrand is odd so

$$\underline{= 0}$$

$$I_{yy} - I_{xx} = \bar{\rho} \int d^3x \left(x^2 - y^2 \right)$$

$$= \bar{\rho} \int_0^h dz \int_0^{r(z)} r'^3 \, dr' \int_0^{2\pi} d\phi \left(\cos^2\phi - \sin^2\phi \right) = 0$$

b) Now consider rotations about P, with

$$\underline{x}_P = (0, 0, z_P)$$

From the generalized parallel axis theorem, we know that rotations around different points generally lead to different parallel axis. However, P is on a principal axis of rotations around the origin, so the principal axes are parallel.

Can also see this by symmetry! The z-axis needs to be a principal axis, and any two linearly independent vectors in the plane orthogonal to it are also degenerate principal axes.

c) Let I_h^o / I_v^o denote the (principal) moments of inertia in the horizontal / vertical directions for rotation about the origin etc.

At the level of tensors,

$$I^P = I^{com} + I^{P*}$$

where I^{P*} is the inertia tensor of the centre of mass with respect to P. Similar holds for P → O, so

$$I^P = (I^O - I^{O*}) + I^{P*}$$

Then we just need to compute the relative moments of inertia. Let $\underline{R}_{com} = (0, 0, z_{com})$ be the centre of mass position from the origin. Then

$$I_{ij}^{O*} = M \left(|\underline{R}|^2 \delta_{ij} - R_i R_j \right)$$

$$= M z_{com}^2 \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right]_{ij} = M z_{com}^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij}$$

The inertia tensor of the centre of mass about $\underline{x}_p$ are

$$I_{ij}^{pt} = M \left(|\underline{R} - \underline{x}_p|^2 \delta_{ij} - (\underline{R} - \underline{x}_p) \cdot (\underline{R} - \underline{x}_p) \right)$$

$$= M (z_{com} - z_p)^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij}$$

hence,

$$I_v^p = I_{zz}^p = I_{zz}^0 = \underline{I_v^0}$$

i.e. nothing changes, while

$$I_h^p = I_{xx}^p = I_{xx}^0 - M z_{com}^2 + M (z_{com} - z_p)^2$$

$$= \underline{I_h^0 - 2M z_{com} z_p + M z_p^2}$$

d) We will assume $I_h^{com} > I_v^{com}$. For rotations to be totally symmetric we essentially want to tune z_p so that

$$I_h^p = I_v^p$$

At $z_p = z_{com}$, $I_h^{com} > I_v^{com}$, but as we move away from the centre of mass, i.e. as $|z_p - z_{com}|$ increases, I_v^p remains the same while I_h^p increases, so this is not possible.

More precisely, taking $p = com$, we know that

$$I_v^{com} = I_v^0$$

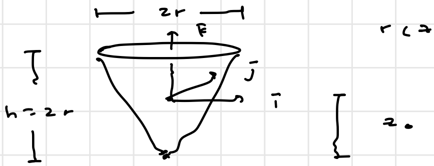
$$I_h^{com} = I_h^0 - M z_{com}^2$$

so we need to solve

$$I_v^{P} = I_v^{com} = I_h^{com} + M(z_{com} - z_P)^2 = I_h^{P}$$

for z_P. This has no real solutions!

2. Ice cream cone of mass M and uniform density



$$r(z) = \frac{z + z_0}{2}$$

Its volume is

$$V = \frac{\pi h^3}{12}$$

a) Choosen centre of mass to be the origin, so in these coordinates

$$\underline{R} = \int_{cone} d^3x\, \rho(\underline{x})\, \underline{r} = \bar{\rho} \int d^3x\, \underline{r} = 0$$

In particular, the z-component is

$$0 = \int_{cone} d^3x\, z$$

can't change these without changing integrand too!

$$= \int_{-z_0}^{h-z_0} dz \int_0^{2\pi} d\phi \int_0^{r(z)} r\, dr \quad z$$

$$= 2\pi \int_{-z_0}^{h-z_0} dz \left[\frac{r^2}{2}\right]_0^{r(z)} z = \pi \int_{-z_0}^{h-z_0} dz \left(\frac{z+z_0}{2}\right)^2 z$$

$$= \frac{\pi}{4}\left[\frac{z^4}{4} + \frac{2z_0 z^3}{3} + \frac{z_0^2 z^2}{2}\right]_{-z_0}^{h-z_0} = \frac{\pi h^3}{48}(3h - 4z_0)$$

so

$$z_0 = \frac{3}{4}h$$

b) Once again we have cylindrical symmetry about the z-axis, so that's a principal axis of inertia. The x- and y-axes (and any linear combination of them) are also (degenerate) principal axes of inertia. Hence, the inertia tensor is diagonal, and $I_{xx} = I_{yy}$. We may therefore write

$$I = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_1 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

c) Just compute. Since we have uniform density

$$I_{ij} = \int_{\text{body}} d^3x \; \rho(\underline{x}) \left(|\underline{x}|^2 \delta_{ij} - x_i x_j \right)$$

$$= \rho \int_{\text{cone}} d^3x \left(|\underline{x}|^2 \delta_{ij} - x_i x_j \right)$$

We can write this in terms of

$$J_{ij} = \int_{\text{cone}} d^3x \; x_i x_j$$

By isotropy

$$= \alpha \, \delta_{ij}$$

and

$$J^i{}_i = \int_{\text{cone}} d^3x \; |\underline{x}|^2$$

careful about limits again!

$$= \int_{-z_0}^{h-z_0} dz \int_0^{2\pi} d\phi \int_0^{r(z)} r\,dr \left(r^2 + z^2 \right)$$

$$= 2\pi \int_{-z_0}^{h-z_0} dz \left[\frac{r^4}{4} + \frac{r^2}{2} z^2 \right]_0^{(z+z_0)/2}$$

$$= 2\pi \int_{-z_0}^{h-z_0} dz \left[\frac{1}{4} \left(\frac{z+z_0}{2} \right)^4 + \frac{1}{2} z^2 \left(\frac{z+z_0}{2} \right)^2 \right]$$

$$= 2\pi \left[\frac{1}{2^6} \frac{(z+z_0)^5}{5} + \frac{1}{2^3} \left(\frac{z^5}{5} + \frac{z^4 z_0}{2} + \frac{z_0^2 z^3}{3} \right) \right]_{-z_0}^{h-z_0}$$

$$= 2\pi \left(\frac{1}{5 \cdot 2^6} \cdot \left(\frac{h}{4}\right)^5 + \frac{1}{2^3} \frac{h^5}{80} \right) = \frac{3\pi h^5}{320}$$

simultaneously,

$$J^i{}_i = 3\alpha \longrightarrow \alpha = \frac{\pi h^5}{320}$$

then

$$I_{ij} = \rho \cdot \left(J^k{}_k \delta_{ij} - J_{ij} \right)$$

$$= \rho \left(3\alpha - \alpha \right) \delta_{ij} = 2\alpha\rho\, \delta_{ij}$$

so the body is fully symmetric. In particular,

$$I = I_1 = I_3 = 2\alpha\rho = 2 \cdot \frac{\cancel{\pi} h^5}{320} \cdot \frac{M}{\cancel{\pi} h^3 / 12}$$

$$= \underline{\frac{3Mh^2}{40}}$$

d) The cone is spinning about an axis through the centre of mass at $\underline{\omega}(t)$

i) Are already in coordinates with principal axis coordinates, so

$$T = \frac{1}{2} \sum_i I_i \omega_i^2 = \frac{1}{2} I \sum_i |\underline{\omega}|^2 = \underline{\frac{3Mh^2}{40} |\underline{\omega}|^2}$$

ii) There are no external forces, so that

$$\left.\frac{d\underline{L}}{dt}\right|_I = \underline{\tau} = 0$$

We pick the rotating frame to be the original coordinate systems with principal axes. Then since $I \propto \delta$, $\underline{\omega} \propto \underline{L}$

$$= \left.\frac{d\underline{L}}{dt}\right|_R + \cancel{\underline{\omega} \times \underline{L}} = \left.\frac{d\underline{L}}{dt}\right|_R$$

Moreover, since I is constant

$$= \frac{d}{dt}\left(I \underline{\omega} \right) = I \underline{\dot{\omega}}$$

Since I is invertible, $\underline{\dot{\omega}} = 0$ and $\underline{\omega}$ stays constant.
This also means energy is conserved!