PLAN

Chit chat

Questions

AF: Ang mom. & Lag. mech.

Go through questions

Essay: Mag. fields or relativistic string

ACP Rapid Feedback 3 : Part I

Angular Momentum

Won't repeat story about angular momentum, just need that

$$\underline{L} = I\underline{\omega} \quad , \quad \frac{d\underline{L}}{dt} = \underline{\tau} = \underline{r} \times \underline{F}$$

Rest of question is just linear algebra.

Lagrangian formalism

View action

$$S = \int dt \quad L(x \ldots) = \int d^D x \quad \mathcal{L}(x, \ldots)$$

as the fundamental object. The Lagrangian can be a function of gen. coordinates or other forms of degrees of freedom like fields. For natural systems, there is a quadratic form

$$L = T - V$$

but there are a lot of Lagrangians that don't fall in this category.

Generically, for a single variable

$$L = L(x, \dot{x}, \ldots)$$

So to extremise S, we want

$$\delta S = \int dt \left(\sum_{n=0}^{\infty} \frac{\partial L}{\partial\left(\frac{d^n x}{dt^n}\right)} \frac{d^n \delta x}{dt^n} \right)$$

$$= \int dt \left(\sum_{n=0}^{\infty} (-1)^n \frac{d^n}{dt^n} \left[\frac{\partial L}{\partial\left(\frac{d^n x}{dt^n}\right)} \right] \delta x + \cancel{\text{boundary}} \right) = 0$$

For this to hold for all δx,

$$\sum_{n=0}^{\infty} (-1)^n \frac{d^n}{dt^n}\left(\frac{\partial L}{\partial\left(\frac{d^n x}{dt^n}\right)}\right)$$

$$= \frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial\left(\frac{dx}{dt}\right)}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial\left(\frac{d^2x}{dt^2}\right)}\right) + \dots = 0$$

<u>Example</u>: Magnetic fields

Consider an electrically charged particle. The correct Lagrangian for this turns out to be

$$L = \tfrac{1}{2}m|\dot{\underline{x}}|^2 + q\,\dot{\underline{x}}\cdot\underline{A}(t,\underline{x}) - q\phi(t,\underline{x})$$

Let's check this.

$$\frac{\partial L}{\partial x^i} = q\dot{x}_j \cdot \frac{\partial A^j}{\partial x^i} - q\frac{\partial\phi}{\partial x^i}$$

$$\frac{\partial L}{\partial \dot{x}^i} = m\dot{x}_i + qA_i(t,\underline{x})$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) = m\ddot{x}_i + q\left(\frac{\partial A_i}{\partial t} + \frac{\partial A_i}{\partial x^j}\frac{dx^j}{dt}\right)$$

Then

$$0 = \frac{\partial L}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right)$$

$$= q\dot{x}^j\partial_i A_j - q\partial_i\phi - \left(m\ddot{x}_i + q\dot{A}_i + \partial_j A_i \dot{x}^j\right)$$

and

$$m\ddot{x}_i = -q(\partial_i\phi + \dot{A}_i) + q\dot{x}^j(\partial_i A_j - \partial_j A_i)$$

Now,

$$E_i = -(\partial_i\phi + \dot{A}_i)$$

$$\varepsilon_{ijk}B^k = \varepsilon_{ijk}\varepsilon^{kmn}\partial_m A_n = (\delta^m_i\delta^n_j - \delta^n_j\delta^m_i)\partial_m A_n$$

$$= \partial_i A_j - \partial_j A_i$$

so that

$$m\ddot{\underline{x}} = q(\underline{E} + \underline{v} \times \underline{B})$$

as expected!

Some remarks are in order.

- This is almost manifestly Lorentz invariant. Choosing proper time parametrization, then

$$u^\mu = (1, \dot{\underline{x}}), \qquad A^\mu = (\phi, \underline{A})$$

and

$$L = -q\phi + q\dot{\underline{x}} \cdot \underline{A} = q u_\mu A^\mu$$

Will see more details later.

- This does not have the canonical quadratic form! In particular, the canonical momentum

$$p_i = \frac{\partial L}{\partial \dot{x}^i} = m\dot{x}_i + qA_i$$

is <u>not</u> the same as the mechanical momentum!

<u>Example</u>: Relativistic string

Let's first consider a relativistic particle. Even in the absence of a potential, this is tricky. One might expect that $(c=1)$

$$L = T = (\gamma - 1)m$$

where

$$\gamma = \frac{1}{\sqrt{1-|\dot{\underline{x}}|^2}}$$

however

$$\frac{\partial L}{\partial \dot{x}^i} = -\frac{1}{2} \cdot -2\dot{x}_i \cdot \left(1 - |\dot{\underline{x}}|^2\right)^{-3/2} = \frac{\dot{x}_i}{\left(1-|\dot{\underline{x}}|^2\right)^{3/2}}$$

which is wrong. We actually need

$$L = \frac{-m}{\gamma} = -m\sqrt{1-|\dot{\underline{x}}|^2}$$

which gives the right momentum

$$p_i = \frac{\partial L}{\partial \dot{x}^i} = -m \cdot -2\dot{x}_i \cdot \frac{1}{2\sqrt{1-|\dot{\underline{x}}|^2}} = \gamma m \dot{x}_i$$

the action is

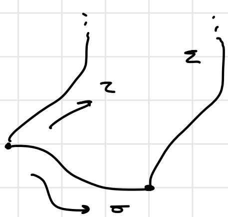
$$S = -m \int dt \left(1 - \delta_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}\right)^{1/2}$$

$$= -m \int \sqrt{dt^2 - \delta_{ij}\, dx^i dx^j} = -m \int \sqrt{-\eta_{\mu\nu}\, dx^\mu dx^\nu}$$

$$= -m \int ds$$

which is the length of the particle worldline.

Now consider a relativistic string. It will sweep out a worldsheet Σ in a d-dim target space



Let $X : \Sigma \to M$ act like coordinates of Σ in the target space. Here the action will just be the area of the worldsheet, i.e.

$$S_{NG} = -T \int_\Sigma dA = -T \int_\Sigma \sqrt{-\det(\gamma)}$$

$$= -T \int d^2\sigma \sqrt{-\det\left(\eta_{\mu\nu} \frac{\partial X^\mu}{\partial \sigma^\alpha} \frac{\partial X^\nu}{\partial \sigma^\beta}\right)}$$

where γ is the induced metric on Σ.

$$= -T \int d^2\sigma \left[(\dot{X} \cdot X')^2 - \dot{X}^2 X'^2 \right]^{1/2}$$

Now,

$$\frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} = -T \cdot \frac{2(\dot{X}\cdot X')X'^\mu - 2X'^2 \dot{X}^\mu}{2\left[(\dot{X}\cdot X')^2 - \dot{X}^2 X'^2\right]^{1/2}} = \frac{(\dot{X}\cdot X')X'^\mu - X'^2 \dot{X}^\mu}{\left[(\dot{X}\cdot X')^2 - \dot{X}^2 X'^2\right]^{1/2}}$$

$$\frac{\partial \mathcal{L}}{\partial X'^\mu} = -T \frac{(\dot{X}\cdot X')\dot{X}^\mu - \dot{X}^2 X'^\mu}{\left[(\dot{X}\cdot X')^2 - \dot{X}^2 X'^2\right]^{1/2}}$$

and the equations of motion will just be

$$\frac{\partial}{\partial \tau}\left[\frac{\partial \mathcal{L}}{\partial \dot{X}^\mu}\right] + \frac{\partial}{\partial \sigma}\left(\frac{\partial \mathcal{L}}{\partial X'^\mu}\right) = 0$$

ACD Problem Sheet 3

1. Consider the orthogonal coordinate system $\{\vec{e}_x, \vec{e}_y, \vec{e}_z\}$ and an object with centre of mass located at the origin with tensor of inertia about the origin

$$I = \frac{1}{2}\begin{pmatrix} I_1 + I_3 & 0 & I_1 - I_3 \\ 0 & 2I_1 & 0 \\ I_1 - I_3 & 0 & I_1 + I_3 \end{pmatrix}$$

a) Just need to find the eigenvalues and -vectors. There's various ways to check this. To diagonalise from scratch, can note that the y-direction is already diagonal - then just need to diagonalise the 2D subspace.

In this case it's easiest to check the eigenvalue equation directly

$$I\,\vec{e}_{1/3} = \frac{1}{2}\begin{pmatrix} I_1 + I_3 & 0 & I_1 - I_3 \\ 0 & 2I_1 & 0 \\ I_1 - I_3 & 0 & I_1 + I_3 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ \pm 1 \end{pmatrix}.$$

$$= \frac{1}{2\sqrt{2}}\begin{pmatrix} I_1 + I_3 \pm (I_1 - I_3) \\ 0 \\ I_1 - I_3 \pm (I_1 + I_3) \end{pmatrix} = \frac{1}{\sqrt{2}}\, I_{1/3}\begin{pmatrix} 1 \\ 0 \\ \pm 1 \end{pmatrix} = \underline{I_{1/3}\,\vec{e}_{1/3}}$$

$$I\,\vec{e}_2 = \frac{1}{2}\begin{pmatrix} I_1 + I_3 & 0 & I_1 - I_3 \\ 0 & 2I_1 & 0 \\ I_1 - I_3 & 0 & I_1 + I_3 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ I_1 \\ 0 \end{pmatrix}$$

$$= \underline{I_2\,\vec{e}_2}$$

So $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ are a set of principal axes with moments I_1, I_2, and I_3 respectively.

b) Two moments about the center of mass are equal, so this is symmetric.

c) Now, the body is spinning freely with

$$\underline{\omega} = \omega_1(t)\,\underline{\hat{e}}_1 + \omega_2(t)\,\underline{\hat{e}}_2 + \omega_3(t)\,\underline{\hat{e}}_3$$

In the absence of external forces, in terms of the principal axes

$$0 = \underline{\tau} = \frac{d\underline{L}}{dt}\Big|_I = \frac{d\underline{L}}{dt}\Big|_* + \underline{\omega} \times \underline{L}$$

$$= \frac{d(I\underline{\omega})}{dt}\Big|_* + \underline{\omega} \times \underline{L}$$

$$= I\dot{\underline{\omega}} + \underline{\omega} \times (I\underline{\omega})$$

In the diagonalized frame

$$\underline{\omega} \times I\underline{\omega} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \times \begin{pmatrix} I_1\omega_1 \\ I_1\omega_2 \\ I_3\omega_3 \end{pmatrix} = \begin{pmatrix} (I_3 - I_1)\,\omega_2\omega_3 \\ (I_1 - I_3)\,\omega_1\omega_3 \\ 0 \end{pmatrix}$$

The $\underline{\hat{e}}_3$-component of the equations of motion are

$$I_3\,\dot{\omega}_3 = 0$$

so

$$\omega_3 = \text{const.}$$

The other two equations are

$$I_1\dot{\omega}_1(t) + (I_3 - I_1)\,\omega_2(t)\,\omega_3 = 0$$
$$I_1\dot{\omega}_2(t) - (I_3 - I_1)\,\omega_1(t)\,\omega_3 = 0$$

To decouple, take e.g. the derivative of the first equation

$$0 = I_1 \ddot{\omega}_1(t) + (I_3 - I_1)\, \dot{\omega}_2(t)\, \omega_3$$

$$= I_1 \ddot{\omega}_1(t) + \frac{(I_3 - I_1)^2}{I_1} \omega_3^2\, \omega_1(t)$$

so

$$\ddot{\omega}_1(t) + \Omega^2 \omega_1(t) = 0$$

for

$$\Omega = \frac{I_3 - I_1}{I_1} \omega_3 \in \mathbb{R}$$

we can solve this easily as

$$\omega_1(t) = A \sin(\Omega t) + B \cos(\Omega t)$$

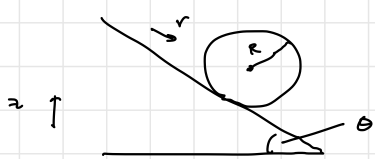
for constant A, B. Then

$$\omega_2(t) = - \frac{I_1 \dot{\omega}_1(t)}{(I_3 - I_1)\omega_3} = \frac{\dot{\omega}_1(t)}{\Omega}$$

$$= A \cos(\Omega t) - B \sin(\Omega t)$$

and A, B are to be determined via initial (boundary) conditions. Note that this is stable.

2. Barrell of mass M and radius R rolls down hill without slipping



convenient coordinate is the distance along the incline. In terms
terms of the height

$$z = -r\sin\theta + \text{const.}$$

and the angular coordinate (radians!)

$$\phi R = r$$

Then

$$L = T - V = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}I\dot{\phi}^2 - mgz$$

$$= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}\cdot\frac{1}{2}m\cancel{R^2}\cdot\left(\frac{\dot{r}}{\cancel{R}}\right)^2 + mgr\sin\theta + \text{const.}$$

$$= \frac{3}{4}m\dot{r}^2 + mgr\sin\theta + \text{const.}$$

Then the Euler - Lagrange equation is

$$0 = \frac{\partial L}{\partial r} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right)$$

$$= mg\sin\theta - \frac{d}{dt}\left(\frac{3}{4}m\cdot 2\dot{r}\right)$$

$$= mg\sin\theta - \frac{3}{2}m\ddot{r}$$

i.e.

$$\ddot{r} = \frac{2}{3}g\sin\theta$$

and

$$r(t) = \frac{1}{3}g t^2\sin\theta + At + B$$

where A, B are constants.