PLAN

Chit chat :

Apologise for marking delay

Too fast - reminder to interrupt (candy)

questions

RF : constraints

Go through questions

Extra :

Infinite derivatives

Hubbard - Stratonovich

Ghost propagator

Ward constraints

Act rapid feedback +

Dynamical systems fully specified by action

$$S = \int dt \; L$$

Sometimes Lagrangian:

$$L = T - V$$

Constraints

Two types:

- Coordinates only: Algebraic
- Also velocities: Differential equations

If explicitly solvable $\longrightarrow$ holonomic!

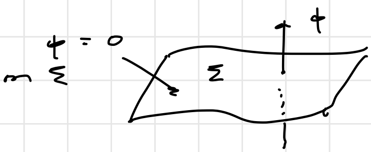
Lagrange multipliers: set of (holonomic or not) constraints

$$\phi_m = 0$$

can enforce by modifying Lagrangian by

$$L \longrightarrow L + \sum_m \lambda_m \phi_m$$

Constraint functions: need regularity conditions. The $\{\phi_m\}$ should form set of coordinates away from constraint surface.



→ $\{\phi_m\}$ are linearly independent on constraint surface!

Example : $\mathbb{R}^3$ with constraints

$$\phi_1 = x^2 + y^2 - 1 \qquad \phi_2 = z^2$$

i.e. an S^1 in the $z = 0$ plane, i.e.

$$\Sigma = \{ \underline{x} \in \mathbb{R}^3 \mid x^2 + y^2 = 1 , \quad z = 0 \}$$

However,

$$\underline{\nabla} \phi_1 = \begin{pmatrix} 2x \\ 2y \\ 0 \end{pmatrix} \qquad \underline{\nabla} \phi_2 = \begin{pmatrix} 0 \\ 0 \\ 2z \end{pmatrix}$$

so $\underline{\nabla} \phi_2 = \underline{0}$ on Σ ! Can do better:

$$\tilde{\phi}_2 = z$$

Conservation laws

Symmetries : transformations that leave the action invariant

→ 1-to-1 correspondence with conservation laws!

Example : Translation invariance

$$\frac{\partial L}{\partial q} = 0 \quad \rightarrow \quad \frac{d}{dt} (p_q) = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) = 0$$

i.e. momentum conservation!

Example: Time translation invariance

$$\frac{\partial L}{\partial t} = 0$$

then

$$\frac{dL}{dt} = \frac{\partial L}{\partial q_i} \dot{q}_i + \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i + \cancel{\frac{\partial L}{\partial t}}$$

$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \right) \dot{q}_i + \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i$$

$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \right)$$

so

$$\frac{d}{dt}\Big(\underbrace{\frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L}_{H} \Big) = 0$$

i.e. energy conservation.

Note: This is a Legendre transformation!

For Hamiltonian formalism, need to map

$$\dot{q}_i \longrightarrow p_i$$

Constraints important when switching to Hamiltonian!

ACP Problem Sheet 4

1.

a–b) consider a system with action

$$S[q] = \int_{t_i}^{t_f} dt \quad L(q(t), q^{(1)}(t), \ldots, q^{(n)}(t); t)$$

varying the action

$$\delta S[q] = \int_{t_i}^{t_f} dt \left[\sum_{m=0}^{n} \frac{\partial L}{\partial q^{(m)}} \delta q^{(m)} \right]$$

can directly manipulate this. more explicitly

$$= \int_{t_i}^{t_f} dt \left[\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial q^{(1)}} \delta q^{(1)} + \ldots + \frac{\partial L}{\partial q^{(n)}} \delta q^{(n)} \right]$$

$$= \int_{t_i}^{t_f} dt \frac{\partial L}{\partial q} \delta q + \left[\frac{\partial L}{\partial q^{(1)}} \delta q \right]_{t_i}^{t_f} - \int_{t_i}^{t_f} dt \frac{d}{dt}\left(\frac{\partial L}{\partial q^{(1)}} \right) \delta q \Bigg]$$

$$+ \left[\frac{\partial L}{\partial q^{(2)}} \delta q^{(1)} \right]_{t_i}^{t_f} - \frac{d}{dt}\left(\frac{\partial L}{\partial q^{(2)}} \right) \delta q \Bigg|_{t_i}^{t_f} + \int_{t_i}^{t_f} dt \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial q^{(2)}} \right) \delta q$$

$$+ \ldots + \left[\frac{\partial L}{\partial q^{(n)}} \delta q^{(n-1)} \right]_{t_i}^{t_f} - \ldots + (-1)^{n-1} \frac{d^{n-1}}{dt^{n-1}} \left(\frac{\partial L}{\partial q^{(n)}} \right) \delta q \Bigg|_{t_i}^{t_f}$$

$$+ (-1)^n \int_{t_i}^{t_f} dt \frac{d^n}{dt^n} \left(\frac{\partial L}{\partial q^{(n)}} \right) \delta q \Bigg]$$

$$= \int_{t_i}^{t_f} dt \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial q^{(1)}} \right) + \ldots + (-1)^n \frac{d^n}{dt^n} \left(\frac{\partial L}{\partial q^{(n)}} \right) \right] \delta q$$

$$+ \frac{\partial L}{\partial q^{(1)}} \delta q + \left[\frac{\partial L}{\partial q^{(2)}} \delta q^{(1)} - \frac{d}{dt}\left(\frac{\partial L}{\partial q^{(2)}} \right) \delta q \right] + \ldots$$

$$+ \left[\frac{\partial L}{\partial q^{(n)}} \delta q^{(n-1)} - \frac{d}{dt}\left(\frac{\partial L}{\partial q^{(n)}} \right) \delta q^{(n-2)} + \ldots + (-1)^{n-1} \frac{d^{n-1}}{dt^{n-1}} \left(\frac{\partial L}{\partial q^{(n)}} \right) \delta q \right]$$

more compactly

$$= \sum_{m=0}^{n} \left[\sum_{k=0}^{m-1} (-1)^k \frac{d^k}{dt^k} \left(\frac{\partial L}{\partial q^{(m)}} \right) \delta q^{(m-k-1)} \right.$$

$$\left. + (-1)^m \int_{t_i}^{t_f} dt \frac{d^m}{dt^m} \left(\frac{\partial L}{\partial q^{(m)}} \right) \delta q \right]$$

The boundary terms vanish, as $\delta q^{(m)} = 0$ for $m = 0, \dots, n-1$

$$= \int_{t_i}^{t_f} dt \sum_{m=0}^{n} (-1)^m \frac{d^m}{dt^m}\left(\frac{\partial L}{\partial q^{(m)}}\right) \delta q$$

For this to hold for all δq, we need

$$\underline{\sum_{m=0}^{n} (-1)^m \frac{d^m}{dt^m}\left(\frac{\partial L}{\partial q^{(m)}}\right) = 0}$$

which are our generalised Euler-Lagrange equations.

c) Consider a system with Lagrangian

$$L = \frac{1}{2}\dot{q}^2 - \frac{1}{2\Lambda^2}\ddot{q}^2$$

Then the Euler-Lagrange equations give

$$0 = \frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}}\right)$$

$$= -\frac{d}{dt}\left(\dot{q}\right) + \frac{d^2}{dt^2}\left(-\frac{1}{\Lambda^2}\ddot{q}\right)$$

$$= -q^{(2)} - \frac{1}{\Lambda^2}q^{(4)}$$

d) Can use a Lagrange constraint or auxiliary field

$$L = \frac{1}{2}\dot{q}^2 - \frac{1}{2\Lambda^2}\dot{Q}^2 + \lambda(Q - \dot{q})$$

First solve the constraint

$$0 = \frac{\delta S}{\delta \lambda} = Q - \dot{q} \longrightarrow Q = \dot{q}$$

Then the auxiliary field

$$0 = \frac{\delta S}{\delta Q} = \frac{\partial L}{\partial Q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Q}}\right)$$

$$= \lambda - \frac{d}{dt}\left(-\frac{1}{\Lambda^2}\dot{Q}\right) = \lambda + \frac{1}{\Lambda^2}\ddot{Q}$$

hence

$$\lambda = -\frac{1}{\Lambda^2}\ddot{Q} = -\frac{1}{\Lambda^2} q^{(3)}$$

Finally,

$$0 = \frac{\delta S}{\delta q} = \frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)$$

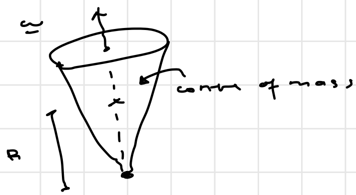
$$= 0 - \frac{d}{dt}(\dot{q} - \lambda) = -\ddot{q} + \dot{\lambda}$$

$$= -q^{(2)} - \frac{1}{\Lambda^2} q^{(4)}$$

as before!

Note: The higher-derivative action field has two normal degrees of freedom – one with the wrong-sign kinetic term, i.e. a ghost.

2. Symmetric top of mass M and principal moments of inertia $I_1 = I_2$, and I_3



with Lagrangian

$$L(\varphi, \dot{\varphi}, \theta, \dot{\theta}, \psi, \dot{\psi}) = \frac{1}{2} I_1 (\dot{\varphi}^2 \sin^2\theta + \dot{\theta}^2)$$

$$+ \frac{1}{2} I_3 (\dot{\psi} + \dot{\varphi}\cos\theta)^2 - MgR\cos\theta$$

a) Euler - Lagrange equations

$$0 = \frac{\partial L}{\partial \varphi} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) = -\frac{d}{dt}\left[I_1 \dot{\varphi}\sin^2\theta + I_3 \cos\theta (\dot{\psi} + \dot{\varphi}\cos\theta)\right]$$

$$0 = \frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right)$$

$$= I_1 \dot\varphi^2 \cdot \sin\theta \cos\theta + I_3 (\dot\psi + \dot\varphi \cos\theta) \cdot -\dot\varphi \sin\theta$$

$$+ mgR \sin\theta - \frac{d}{dt}\left(I_1 \dot\theta \right)$$

$$0 = \frac{\partial L}{\partial \psi} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\psi}\right) = -\frac{d}{dt}\left[I_3 (\dot\psi + \dot\varphi \cos\theta)\right]$$

b) conserved quantities.

The obvious conserved quantities are the conjugated momenta due to cyclic coordinates

$$p_\psi = \frac{\partial L}{\partial \dot\psi} = I_3 (\dot\psi + \dot\varphi \cos\theta)$$

$$p_\varphi = \frac{\partial L}{\partial \dot\varphi} = I_1 \dot\varphi \sin^2\theta + I_3 \cos\theta (\dot\psi + \dot\varphi \cos\theta)$$

$$= I_1 \dot\varphi \sin^2\theta + \cos\theta \, p_\psi$$

for which by the eq. of motion

$$\frac{dp_\psi}{dt} = 0 \qquad \frac{dp_\varphi}{dt} = 0$$

Here, the corresponding symmetry is invariance under translation of φ and ψ.

We also have no explicit time dependence, i.e. time translation symmetry, so

$$H = \sum_i p_i \dot q_i - L$$

$$= p_\varphi \dot\varphi + p_\theta \dot\theta + p_\psi \dot\psi - L$$

$$= (\underbrace{I_1 \dot\varphi \sin^2\theta} + \underline{\cos\theta \, p_\psi}) \dot\varphi + \underline{I_1 \dot\theta} \cdot \dot\theta$$

$$+ \underline{p_\varphi \dot\psi} - \frac{1}{2} I_1 (\underline{\dot\varphi^2 \sin^2\theta} + \underline{\dot\theta^2}) - \underline{\frac{1}{2} I_3 (\dot\psi + \dot\varphi \cos\theta)^2}$$

$$+ MgR\cos\theta$$

$$= \frac{1}{2} I_1 \left(\sin^2\theta \, \dot\varphi^2 + \dot\theta^2 \right) + \frac{1}{2} I_3 (\dot\psi + \dot\varphi \cos\theta)^2$$

$$+ MgR\cos\theta$$

For Hamiltonian formalism, map $\dot q_i \to p_i$. Recall

$$p_\varphi = I_1 \dot\varphi \sin^2\theta + \cos\theta \, p_\psi$$

$$p_\psi = I_3 (\dot\psi + \dot\varphi \cos\theta)$$

$$p_\theta = I_1 \dot\theta$$

Then

$$H = \frac{1}{2} I_1 \sin^2\theta \left(\frac{\cos\theta \, p_\psi - p_\varphi}{I_1 \sin^2\theta} \right)^2 + \frac{1}{2} I_1 \left(\frac{p_\theta}{I_1} \right)^2$$

$$+ \frac{1}{2} I_3 \left(\frac{p_\psi}{I_3} \right)^2 + MgR\cos\theta$$

$$= \frac{1}{2} \frac{(\cos\theta \, p_\psi - p_\varphi)^2}{I_1 \sin^2\theta} + \frac{1}{2} \frac{p_\theta^2}{I_1} + \frac{1}{2} \frac{p_\psi^2}{I_3} + MgR\cos\theta$$

$$= \underline{T + V}$$