PLAN

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Normal mode

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ACP Rapid Feedback 5

Normal modes

Natural system: kinetic term quadratic in derivatives

$$T = \frac{1}{2} A_{ij} \dot{q}^i \dot{q}^j \quad , \quad i \in \{1, \dots, D\}$$

choose coordinates $q^i \to q^i$ where $A \to \delta$

$$L = \frac{1}{2} \delta_{ij} \dot{q}^i \dot{q}^j - V(q)$$

Equations of motion given by E-L:

$$0 = \frac{\partial L}{\partial q^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = -\frac{\partial V}{\partial q^i} - \ddot{q}_i$$

$$\underline{\ddot{q}^i = -\frac{\partial V}{\partial q_i}}$$

Point $q = q_0$ is equilibrium when

$$\ddot{q}^i (q_0) = -\left.\frac{\partial V}{\partial q_i}\right|_{q_0} = 0$$

Small fluctuations around equilibrium

$$q = q_0 + \delta q$$

Taylor expand V around q_0

$$V(q = q_0 + \delta q)$$

$$= V(q_0) + \cancel{\left.\frac{\partial V}{\partial q^i}\right|_{q_0}} \delta q^i + \frac{1}{2} \left.\frac{\partial^2 V}{\partial q^i \partial q^j}\right|_{q_0} \delta q^i \delta q^j$$

$$+ \dots$$

so Lagrangian for small fluctuations around q_0

$$L = \frac{1}{2}\delta\dot{q}^i\delta\dot{q}^j - \frac{1}{2}\underbrace{\frac{\partial^2 V}{\partial q^i \partial q^j}\Big|_{q_0}}_{\text{lectures: } k_{ij}}\delta q^i \delta q^j + O(\delta q^3)$$

Equations of motion are

$$\delta\ddot{q}_i = -\frac{\partial^2 V}{\partial q^i \partial q^j}\Big|_{q_0}\delta q^j$$

second-order, constant coefficients - like damped harmonic oscillator

$$\delta q_i = A_i e^{i\omega t}$$

for constants $c_i \in \mathbb{C}$. plugging in

$$-\omega^2 A_i = -\frac{\partial^2 V}{\partial q^i \partial q^j}\Big|_{q_0} A_j$$

$\to$ eigenvalue problem: k_{ij} symmetric so D real eigenvalues

labels eigenvalue

$$\omega^2_\alpha \quad , \quad \alpha \in \{1, \dots, D\}$$

and D orthogonal eigenvectors

$$v^i_\alpha = \begin{pmatrix} v^1_\alpha \\ \vdots \\ v^D_\alpha \end{pmatrix}$$

General solution for each mode $(\pm\omega_\alpha)$

$$\delta q^i_\alpha = \left(A_\alpha e^{i\omega_\alpha t} + B_\alpha e^{-i\omega_\alpha t}\right) v^i_\alpha$$

and generally

$$\delta q^i = \sum_\alpha \delta q^i_\alpha = \sum_\alpha \left(A_\alpha e^{i\omega_\alpha t} + B_\alpha e^{-i\omega_\alpha t}\right) v^i_\alpha$$

→ Instabilities associated to $\omega_\alpha^2 < 0$.

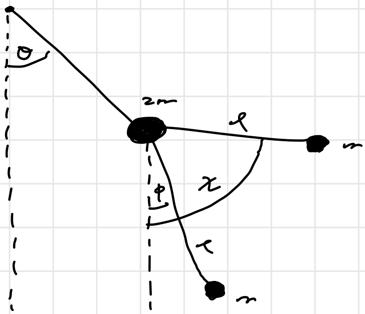
This has 2D free constants - as expected for D 2nd order DEs.

corollary: Initial conditions given by normal mode, i.e. $A_\alpha, B_\alpha \neq 0$ for single α, then never stop normal mode

→ "decoupled" modes of motion!

ACP Problem sheet 5

1. Triple pendulum



For small angles : $|\theta|, |\phi|, |\chi| \ll 1$, the Lagrangian is

$$L = \frac{1}{2} m l^2 \left(2\dot{\theta} + (\dot{\theta} + \dot{\phi})^2 + (\dot{\phi} + \dot{\chi})^2 \right) - \frac{1}{2} g m l \left(4\theta^2 + \phi^2 + \chi^2 \right)$$

a) only energy is obviously conserved, i.e.

$$H = T + V$$

$$= \frac{1}{2} m l^2 \left(2\dot{\theta}^2 + (\dot{\theta} + \dot{\phi})^2 + (\dot{\theta} + \dot{\chi})^2 \right) + \frac{1}{2} g m l \left(4\theta^2 + \phi^2 + \chi^2 \right)$$

b) canonically normalise

$$\begin{pmatrix} q^1 \\ q^2 \\ q^3 \end{pmatrix} = \begin{pmatrix} (2ml^2)^{1/2} & \theta \\ (ml^2)^{1/2} & (\theta + \phi) \\ (ml^2)^{1/2} & (\theta + \chi) \end{pmatrix} = (ml^2)^{1/2} \underbrace{\begin{pmatrix} \sqrt{2} & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}}_{M} \begin{pmatrix} \theta \\ \phi \\ \chi \end{pmatrix}$$

By inverting

$$\begin{pmatrix} \theta \\ \phi \\ \chi \end{pmatrix} = (ml^2)^{-1/2} \underbrace{\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 1 \end{pmatrix}}_{M^{-1}} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$$

so that

$$\frac{1}{2} g m \ell \left(\theta^2 + \phi^2 + x^2 \right)$$

$$= \frac{1}{2} g m \ell \cdot (m\ell^2)^{-1} \left[4 \left(\frac{q_1}{\sqrt{2}}\right)^2 + \left(-\frac{1}{\sqrt{2}} q_1 + q_2 \right)^2 + \left(-\frac{1}{\sqrt{2}} q_1 + q_3 \right)^2 \right]$$

$$= \frac{1}{2} \frac{g}{\ell} \left(4 \cdot \frac{1}{2} q_1^2 + \frac{1}{2} q_1^2 - \frac{2}{\sqrt{2}} q_1 q_2 + q_2^2 + \frac{1}{2} q_1^2 - \frac{2}{\sqrt{2}} q_1 q_3 + q_3^2 \right)$$

$$= \frac{1}{2} \frac{g}{\ell} \left(3 q_1^2 + q_2^2 + q_3^2 - \sqrt{2}\, q_1 (q_2 + q_3) \right)$$

i.e. we have

$$L = \frac{1}{2} \delta_{ij} \dot{q}^i \dot{q}^j - \frac{1}{2} k_{ij} q^i q^j$$

with

symmetry on off-diagonals

$$k = \frac{g}{\ell} \begin{pmatrix} 3 & -\sqrt{2}/2 & -\sqrt{2}/2 \\ \cdot & 1 & 0 \\ \cdot & \cdot & 1 \end{pmatrix}$$

check: change of coordinates on derivatives

$$\frac{\partial}{\partial q^1} = \frac{\partial \theta}{\partial q^1} \frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial q^1} \frac{\partial}{\partial \phi} + \frac{\partial x}{\partial q^1} \frac{\partial}{\partial x}$$

$$= (m\ell^2)^{-1/2} \cdot \frac{1}{\sqrt{2}} \left[\frac{\partial}{\partial \theta} - \frac{\partial}{\partial \phi} - \frac{\partial}{\partial x} \right]$$

$$\frac{\partial}{\partial q^2} = \frac{\partial \theta}{\partial q^2} \frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial q^2} \frac{\partial}{\partial \phi} + \frac{\partial x}{\partial q^2} \frac{\partial}{\partial x}$$

$$= (m\ell^2)^{-1/2} \cdot \frac{\partial}{\partial \phi}$$

Then

$$k_{12} = \frac{\partial^2 V}{\partial q^1 \partial q^2} = \frac{\partial}{\partial q^1} \left((m\ell^2)^{-1/2} \cdot -mg\ell\phi \right)$$

$$= -(m\ell^2)^{-1/2} \cdot \frac{1}{\sqrt{2}} \cdot mg\ell = \underline{-\frac{g}{\ell} \cdot \frac{1}{\sqrt{2}}}$$

c) Euler-Lagrange equations

$$0 = \frac{\partial L}{\partial q^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) = -k_{ij} q^j - \frac{d}{dt}\left(\dot{q}_i\right)$$

i.e.

$$\ddot{q}^i + k_{ij} q^j = 0$$

i) Trivial solution: The state

$$q^i_t = 0$$

is a solution since

$$\ddot{q}^i_t = 0 \quad , \quad k_{ij} q^j_t = 0$$

In terms of the old coordinates

$$\begin{pmatrix} \theta \\ \phi \\ x \end{pmatrix} = (me^2)^{-1/2} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

trivially since this is the null vector.

ii) non-trivial solution: want to find solution with $\theta, \dot{\theta} = 0$. Generally

$$q_0 = \begin{pmatrix} q^1 \\ q^2 \\ q^3 \end{pmatrix} = (me^2)^{1/2} \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \phi \\ x \end{pmatrix} = (me^2)^{1/2} \begin{pmatrix} 0 \\ \phi \\ x \end{pmatrix}$$

can't just set $\theta = 0$ in L!

so $q^1, \dot{q}^1 = 0$.

Check if normal mode

$$K \cdot q_0 = \frac{g}{\ell}\begin{pmatrix} 3 & -\sqrt{2}/2 & -\sqrt{2}/2 \\ \cdot & 1 & 0 \\ \cdot & \cdot & 1 \end{pmatrix}\begin{pmatrix} 0 \\ q^2 \\ q^3 \end{pmatrix} = \frac{g}{\ell}\begin{pmatrix} -\frac{1}{\sqrt{2}}(q^2+q^3) \\ q^2 \\ q^3 \end{pmatrix}$$

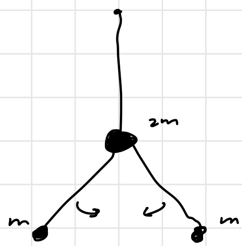
i.e. need

$$\underline{q_0^2 + q_0^3 = 0} \longrightarrow \underline{\phi + x = 0}$$

with eigenvalue

$$\underline{\omega^2 = g/\ell} \rightarrow \text{stable oscillations}$$

physically:



Note: there's other modes too. To find eigenvalues, generally solve. Scale $\lambda = \frac{g}{\ell}\tilde{\lambda}$

$$\det\left(K - \frac{g}{\ell}\tilde{\lambda} I\right) = \left(\frac{g}{\ell}\right)^3 \det \begin{vmatrix} 3-\tilde{\lambda} & -1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{2} & 1-\tilde{\lambda} & 0 \\ -1/\sqrt{2} & 0 & 1-\tilde{\lambda} \end{vmatrix}$$

$$= \left(\frac{g}{\ell}\right)^3 \left[(3-\tilde{\lambda})(1-\tilde{\lambda})^2 - \left(-\frac{1}{\sqrt{2}}\right) \cdot -\frac{1}{\sqrt{2}}(1-\tilde{\lambda}) - \frac{1}{\sqrt{2}} \cdot -\left(-\frac{1}{\sqrt{2}}\right)(1-\tilde{\lambda}) \right]$$

$$= \left(\frac{g}{\ell}\right)^3 (1-\tilde{\lambda}) \left[(3-\tilde{\lambda})(1-\tilde{\lambda}) - \frac{1}{2} - \frac{1}{2} \right]$$

$$= \left(\frac{g}{\ell}\right)^3 (1-\tilde{\lambda}) \left(\tilde{\lambda}^2 - 4\tilde{\lambda} + 2 \right)$$

We already know $\tilde{\lambda}_0 = 1$. The other two are

$$\lambda_\pm = \frac{+4 \pm \sqrt{16 - 4 \cdot 2}}{2} = \frac{4 \pm 2\sqrt{2}}{2} = \underline{2 \pm \sqrt{2}}$$

check eigenmodes.

$$K\underline{q} = \frac{g}{\ell} \begin{pmatrix} 3 & -1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{2} & 1 & 0 \\ -1/\sqrt{2} & 0 & 1 \end{pmatrix} \begin{pmatrix} q^1 \\ q^2 \\ q^3 \end{pmatrix} = \frac{g}{\ell} \begin{pmatrix} 3q^1 - \frac{1}{\sqrt{2}}(q^2 + q^3) \\ -\frac{1}{\sqrt{2}} q^1 + q^2 \\ -\frac{1}{\sqrt{2}} q^1 + q^3 \end{pmatrix}$$

$$= (2 \pm \sqrt{2}) \frac{g}{\ell} \begin{pmatrix} q^1 \\ q^2 \\ q^3 \end{pmatrix} = \lambda_\pm \underline{q}$$

Can scale $q^1 = 1$, so that

$$-\frac{1}{\sqrt{2}} + q^2 = (2 \pm \sqrt{2}) q^2$$

$$(1 \pm \sqrt{2}) q^2 = -\frac{1}{\sqrt{2}}$$

$$q^2 = -\frac{1}{\sqrt{2}(1 \pm \sqrt{2})} = -\frac{1}{(\pm 2 + \sqrt{2})}$$

and q^3 satisfies same equation, so

$$q^3 = q^2$$

More sensible to scale $q^2 = q^3 = 1$, so

$$\underline{q}_\pm = \begin{pmatrix} -\sqrt{2} \mp 2 \\ 1 \\ 1 \end{pmatrix}$$

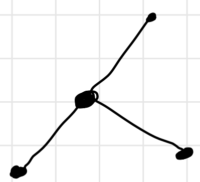
in terms of old coordinates

$$\underline{v} \propto \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 1 \end{pmatrix} \begin{pmatrix} -\sqrt{2} \mp 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \mp \sqrt{2} \\ 1 \pm \sqrt{2} + 1 \\ 1 \pm \sqrt{2} + 1 \end{pmatrix}$$

$$= (-1 \mp \sqrt{2}) \begin{pmatrix} 1 \\ \frac{2 \pm \sqrt{2}}{-1 \mp \sqrt{2}} \\ \frac{2 \pm \sqrt{2}}{-1 \mp \sqrt{2}} \end{pmatrix} \propto \begin{pmatrix} 1 \\ \mp \sqrt{2} \\ \mp \sqrt{2} \end{pmatrix}$$

i.e. the double pendulum modes

Note: starting in normal mode will leave me in normal mode, however generic initial conditions e.g.



is linear combination of the modes - will all evolve separately!

2. Standard action

$$S[\phi] = \int dt \; L(\phi)$$

$$= \int dt \, dx \left(\frac{1}{2} \dot{\phi}^2 - \frac{1}{2} \phi'^2 - V(\phi) \right)$$

has class. equations of motion

$$0 = \frac{\partial \mathcal{L}}{\partial \phi} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \right) - \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial \phi'} \right)$$

$$= -V' - \frac{d}{dt}\left(\dot{\phi} \right) - \frac{d}{dx}\left(-\phi' \right)$$

$$= -\ddot{\phi} + \phi'' - V'(\phi)$$

Note: could have written covariantly

$$\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\phi'^2 = -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi$$

and used

$$0 = \frac{\partial\mathcal{L}}{\partial\phi} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\phi}{\partial x^\mu}\right)}\right)$$

Now consider

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 + \dot{\phi}\phi' - \frac{1}{2}\phi'^2 - V(\phi)$$

then

$$0 = \frac{\partial\mathcal{L}}{\partial\phi} - \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\right) - \frac{d}{dx}\left(\frac{\partial\mathcal{L}}{\partial\phi'}\right)$$

$$= -V'(\phi) - \frac{d}{dt}\left(\dot{\phi} + \underline{\phi'}\right) - \frac{d}{dx}\left(\underline{\dot{\phi}} - \phi'\right)$$

$$= -\ddot{\phi} - 2\dot{\phi}' + \phi'' - V'(\phi)$$

Note: This violates symmetry under

$$P : x \mapsto -x \quad , \qquad T : t \to -t$$

separately, but not together:

$$PT : (t, x) \mapsto (-t, -x)$$

In nature: can have violations of C, P, T individually and pairwise

- P : Reverse space, i.e. $\underline{x} \mapsto -\underline{x}$.
- T : Reverse time, i.e. $t \to -t$.

- C: conjugate "charges", i.e. send particle to antiparticle.
→ must be broken due to matter-antimatter imbalance!

weak force violates C, P (Neutrinos), and CP (Kaons, Nobel 1980).

CPT theorem: CPT is symmetry of any Lorentz inv. and local QFT with Hermitian Hamiltonian.

$$R_1^2 = 2R_2^2$$