ACP: Rapid feedback 6

Hamiltonian Mechanics: Symplectic Geometry

Why care?

- historical development of QM
- integrable systems
- symplectic geometry ⟵ concentrate on this!

Disclaimer: new perspective, not expected to be familiar with it!

Start from Lagrangian

$$L = L(q^i, \dot{q}^i), \qquad i \in \{1, \dots, n\}$$

If the canonical momenta

$$\frac{\partial L}{\partial \dot{q}^i},$$

are independent, i.e.

$$\det\left(\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\right) \neq 0$$

then we define the Hamiltonian to be the Legendre transform of this (otherwise constraint analysis!)

$$H(q,p) = \frac{\partial L}{\partial \dot{q}^i} \dot{q}^i - L(q, \dot{q})$$

$$= p_i \dot{q}^i (q,p) - L[q, \dot{q}(q,p)]$$

Form of equations easy in this.

Consider the (q, p) - derivatives

$$\frac{\partial H}{\partial p_i} = \dot{q}^i(q,p) + p_j \frac{\partial \dot{q}^j(q,p)}{\partial p_i} - \frac{\partial L(q,\dot{q})}{\partial \dot{q}^j} \frac{\partial \dot{q}^j(q,p)}{\partial p_i}$$

$$= \underline{\underline{\dot{q}^i}}$$

$$\frac{\partial H}{\partial q^i} = p_j \cdot \frac{\partial \dot{q}^j(q,p)}{\partial q^i} - \left(\frac{\partial L(q,\dot{q})}{\partial q^i} + \frac{\partial L(q,\dot{q})}{\partial \dot{q}^j} \frac{\partial \dot{q}^j(q,p)}{\partial q^i} \right)$$

$$= - \frac{\partial L}{\partial q^i} = - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i} \right) = \underline{\underline{- \dot{p}_i}}$$

→ <u>Hamilton's equations</u>

Idea: <u>phase space</u> coordinates

$$y^\mu = (q^i, p^j), \quad \mu \in \{1, \ldots, 2n\}$$

Hamilton's equations are equivalent to

$$\frac{dy^\mu}{dt} = \Omega^{\mu\nu} \frac{\partial H}{\partial y^\nu}$$

where

$$\Omega = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}$$

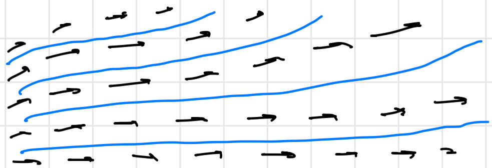
is the <u>symplectic 2-form</u> in local canonical coordinates.

$$\Omega^2 = -1, \quad \Omega_{\alpha\beta} = -\Omega_{\beta\alpha}$$

→ Hamilton's equations are first-order equations for trajectory/<u>flow</u>.

Defines flow of functions. Given generator G and parameter λ

$$\frac{dy^\mu}{d\lambda} = \Omega^{\mu\nu} \partial_\nu G$$



infinitesimally

$$\delta q^i = \frac{\partial G}{\partial p_i} \delta\lambda \qquad \delta p_i = -\frac{\partial G}{\partial q^i} \delta\lambda$$

For any other function F

$$\frac{dF}{d\lambda} = \frac{\partial F}{\partial \lambda} + \frac{\partial F}{\partial y^\mu} \frac{dy^\mu}{d\lambda} = \frac{\partial F}{\partial \lambda} + \Omega^{\mu\nu} \partial_\mu F \, \partial_\nu G = \frac{\partial F}{\partial \lambda} + \{F, G\}$$

where, in canonical coordinates, the Poisson structure / bracket is

$$\{F, G\} = \Omega^{\mu\nu} \partial_\mu F \, \partial_\nu G = \frac{\partial F}{\partial q^i} \frac{\partial G}{\partial p_i} - \frac{\partial G}{\partial p_i} \frac{\partial F}{\partial q^i}$$

→ G generates change in F.

Example: Canonical Poisson brackets

$$\{y^\mu, y^\nu\} = \Omega^{\alpha\beta} \partial_\alpha y^\mu \partial_\beta y^\nu = \Omega^{\mu\nu}$$

or more explicitly,

$$\{q^i, q^j\} = 0 = \{p^i, p^j\}$$

$$\{q^i, p^j\} = \delta^{ij}$$

Example: Hamilton's equations

$$\frac{dy^\mu}{dt} = \Omega^{\mu\nu} \partial_\nu H$$

more explicitly:

$$\dot{q}^i = \{q^i, H\} = \frac{\partial q^i}{\partial q^j}\frac{\partial H}{\partial p_j} - \frac{\partial q^i}{\partial p_j}\frac{\partial H}{\partial q^j} = \frac{\partial H}{\partial p_i}$$

$$\dot{p}_i = \{p_i, H\} = \frac{\partial p_i}{\partial q^j}\frac{\partial H}{\partial p_j} - \frac{\partial p_i}{\partial p_j}\frac{\partial H}{\partial q^j} = -\frac{\partial H}{\partial q^i}$$

→ H generates <u>time translations</u>!

As a corollary

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \{H, H\} = \frac{\partial H}{\partial t}$$

→ H conserved if time translation invariance

This gives us <u>Noether's theorem</u>:

H invariant under G ⟷ G conserved

$\{H, G\} = 0$ ⟷ $\{G, H\} = 0$

<u>Canonical transformations</u>: Transformations M, for which

$$\Omega = M \Omega M^T$$

i.e. symplectic group.

→ equations of motion and Poisson bracket invariant (iff canonical brackets preserved!?).

<u>Example</u>: Non-trivial

$Q^i = p^i$, $\quad P^i = -q^i$

change of coordinates

$$M = \begin{pmatrix} \partial Q^i/\partial q^j & \partial Q^i/\partial p_j \\ \partial P^i/\partial q^j & \partial P^i/\partial p_j \end{pmatrix} = \begin{pmatrix} 0 & \delta^i_j \\ -\delta^i_j & 0 \end{pmatrix} = \Omega$$

so

$$M \Omega M^T = \Omega \Omega \Omega^T = -\Omega^T = \Omega$$

Example (Laplace - Runge - Lenz): Angular momentum

$$L_i = \varepsilon_{ijk} x_j p_k$$

satisfies

$$\{L_i, x_j\} = \varepsilon_{ikl} \{x^k p^l, x_j\} = \varepsilon_{ikl} x^k \{p^l, x_j\}$$

$$= \varepsilon_{ikl} x^k \cdot -\delta^l_j = \varepsilon_{ijk} x^k$$

$$\{L_i, p_j\} = \varepsilon_{ikl} \{x^k p^l, p_j\} = \varepsilon_{ikl} \{x^k, p_j\} p^l$$

$$= \varepsilon_{ikl} \cdot \delta^k_j p^l = \varepsilon_{ijk} p_k$$

$$\{L_i, L_j\} = \varepsilon_{jkl} \{L_i, x^k p^l\}$$

$$= \varepsilon_{jkl} \left(x^k \{L_i, p^l\} + \{L_i, x^k\} p^l \right)$$

$$= \underset{\varepsilon_{ljk}}{\varepsilon_{jkl}} \left(x^k \cdot \underset{\varepsilon_{mil}}{\varepsilon_i{}^l{}_m} p^m + \underset{\varepsilon_{mik}}{\varepsilon_i{}^k{}_m} x^m p^l \right)$$

$$= (\delta_{mj} \delta_{ik} - \delta_{mk} \delta_{ji}) x^k p^m$$

$$+ (\delta_{lm} \delta_{ji} - \delta_{li} \delta_{jm}) x^m p^l$$

$$= x_i p_j - x_k x^k \delta_{ij} + \delta_{ij} x_k x^k - x_j p_i$$

$$= (\delta_{im} \delta_{jn} - \delta_{in} \delta_{jm}) x^m p^n$$

$$= \varepsilon_{ijk} \varepsilon_{mn}{}^k x^m p^n = \varepsilon_{ijk} L^k$$

Furthermore

$$\{L_i, |L|^2\} = \{L_i, L_j L^j\}$$

$$= \{L_i, L_j\} L^j + L^j \{L_i, L_j\}$$

$$= \varepsilon_{ijk} L^k L^j + L^j \varepsilon_{ijk} L^k = 0$$

For central potentials

$$H = \frac{|p|^2}{2m} + V(|x|)$$

then

$$\{L_i, \frac{|p|^2}{2m}\} = \frac{1}{2m} \{L_i, p_j p^j\}$$

$$= \frac{1}{2m} \left(p_j \{L_i, p^j\} + \{L_i, p_j\} p^j \right)$$

$$= \frac{1}{2m} \left(p_j \cdot \varepsilon_i{}^j{}_k p^k + \varepsilon_{ijk} p^k p^j \right) = 0$$

$$\{L_i, V(|x|)\} = \{L_i, x_j x^j\} \frac{1}{2|x|} V'(|x|)$$

$$= \frac{V'(|x|)}{2|x|} \cdot \left(x_j \varepsilon_i{}^j{}_k x^k + \varepsilon_{ijk} x^k \cdot x^j \right) = 0$$

so that

$$\{L_i, H\} = 0$$

→ angular momentum conserved

Now consider the Laplace-Runge-Lenz vector

$$A^i = \varepsilon^i{}_{jk} p^j L^k - mk \frac{x^i}{|x|}$$

For this

$$\{A_i, L_j\} = \varepsilon_{ijk} A^k$$

$$\{A_i, A_j\} = -2m\left(\frac{|p|^2}{2m} - \frac{k}{|x|}\right)\varepsilon_{ijk} L^k$$

when

$$V(|x|) = -\frac{k}{|x|}$$

then

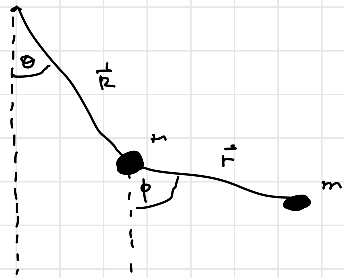
$$\{A_i, A_j\} = -2m\, H\, \varepsilon_{ijk} L^k$$

$$\{A_i, H\} = 0$$

→ A_i also conserved, extra hidden symmetry!

ACP: Problem Sheet 6

1. Double pendulum with $m \neq M$, $r \neq R$



has Lagrangian

$$L = \frac{1}{2} M R^2 \dot{\theta}^2 + \frac{1}{2} m \left[R^2 \dot{\theta}^2 + r^2 \dot{\phi}^2 + 2Rr \dot{\theta}\dot{\phi} \cos(\theta - \phi) \right]$$

$$- MgR(1 - \cos\theta) - mg\left[R(1-\cos\theta) + r(1 - \cos\phi) \right]$$

For small oscillations,

$$\approx \frac{1}{2}(M+m) R^2 \dot{\theta}^2 + \frac{1}{2} m r^2 \dot{\phi}^2 + mRr\dot{\theta}\dot{\phi}$$

$$- \frac{1}{2}(M+m) g r \theta^2 - \frac{1}{2} m g r \phi^2$$

The <u>generalized momenta</u> are

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \partial L / \partial \dot{\theta} \\ \partial L / \partial \dot{\phi} \end{pmatrix} = \begin{pmatrix} (M+m) R^2 \dot{\theta} + mRr\dot{\phi} \\ mr^2 \dot{\phi} + mRr\dot{\theta} \end{pmatrix} = \underbrace{\begin{pmatrix} (M+m)R^2 & mRr \\ mRr & mr^2 \end{pmatrix}}_{M} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Compute

$$\det M = (M+m)R^2 \cdot mr^2 - (MRr)^2 = Mm R^2 r^2 \neq 0$$

and M is invertible

$$\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} = \frac{1}{M m R^2 r^2} \begin{pmatrix} m r^2 & -mRr \\ -mRr & (M+m)R^2 \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \frac{1}{MR}\left(\frac{p_\theta}{R} - \frac{p_\phi}{r}\right) \\ \frac{1}{mr}\left[\left(1+\frac{m}{M}\right)\frac{p_\phi}{r} - \frac{m}{M}\frac{p_\theta}{R}\right] \end{pmatrix}$$

Then, the Hamiltonian is

$$H = p_\theta \dot\theta + p_\phi \dot\phi - L$$

$$= p_\theta \dot\theta + p_\phi \dot\phi - \frac{1}{2}(M+m)R^2\dot\theta^2 - \frac{1}{2} m r^2 \dot\phi^2 - mRr\,\dot\theta\dot\phi$$

$$+ \frac{1}{2}(M+m) g R\,\theta^2 + \frac{1}{2} m g r\,\phi^2$$

$$= p_\theta \cdot \frac{1}{MR}\left(\frac{p_\theta}{R} - \frac{p_\phi}{r}\right) + p_\phi \cdot \frac{1}{mr}\left(\left(1+\frac{m}{M}\right)\frac{p_\phi}{r} - \frac{m}{M}\frac{p_\theta}{R}\right)$$

$$- \frac{1}{2}(M+m)\cancel{R^2} \cdot \frac{1}{(M\cancel{R})^2}\left(\frac{p_\theta}{R} - \frac{p_\phi}{r}\right)^2$$

$$- \frac{1}{2}\cancel{mr^2}\,\frac{1}{m}\frac{1}{\cancel{(mr)^2}}\left(\left(1+\frac{m}{M}\right)\frac{p_\phi}{r} - \frac{m}{M}\frac{p_\theta}{R}\right)^2$$

$$- \cancel{mRr}\,\frac{1}{M\cancel{R}} \cdot \frac{1}{\cancel{mr}}\left(\frac{p_\theta}{R} - \frac{p_\phi}{r}\right)\left(\left(1+\frac{m}{M}\right)\frac{p_\phi}{r} - \frac{m}{M}\frac{p_\theta}{R}\right)$$

$$+ \frac{1}{2}(M+m) g R\,\theta^2 + \frac{1}{2} m g r\,\phi^2$$

$$= \left(\frac{p_\theta}{R}\right)^2\left[\frac{1}{M} - \frac{1}{2}\frac{M+m}{M^2} - \frac{1}{2}\frac{1}{m}\cdot\left(\frac{m}{M}\right)^2 - \frac{1}{M}\cdot -\frac{m}{M}\right]$$

$$+ \left(\frac{p_\phi}{r}\right)^2\left[\frac{1}{m}\left(1+\frac{m}{M}\right) - \frac{1}{2}(M+m)\cdot\frac{1}{M^2} - \frac{1}{2}\cdot\frac{1}{m}\left(1+\frac{m}{M}\right)^2\right.$$

$$\left. - \frac{1}{M}\cdot -\left(1+\frac{m}{M}\right)\right]$$

$$+ \frac{p_\theta}{R}\frac{p_\phi}{r}\cdot\left[-\frac{1}{M} + \frac{1}{m}\cdot -\frac{m}{M} - \frac{1}{2}(M+m)\cdot\frac{1}{M^2}\cdot -2\right.$$

$$\left. - \frac{1}{2}\frac{1}{m}\left(1+\frac{m}{r}\right)\cdot\frac{m}{M}\cdot -2 - \frac{1}{M}\left(1+\frac{m}{M}+\frac{m}{M}\right)\right]$$

$$+ \frac{1}{2}(M+m) g R\,\theta^2 + \frac{1}{2} m g r\,\phi^2$$

$$= \left(\frac{p_\theta}{R}\right)^2 \left(\frac{1}{M} - \frac{1}{2}\frac{1}{M} - \frac{1}{2}\frac{m}{M^2} - \frac{1}{2}\cdot\frac{m}{M^2} + \frac{m}{M^2} \right)$$

$$+ \left(\frac{p_\phi}{r}\right)^2 \left[\frac{1}{m} + \frac{1}{M} - \frac{1}{2}\frac{1}{M} - \frac{1}{2}\frac{m}{M^2} - \frac{1}{2m}\left(1 + 2\frac{m}{M} + \frac{m^2}{M^2}\right) \right.$$

$$\left. + \frac{1}{M} + \frac{m}{M^2} \right]$$

$$+ \frac{p_\theta}{R}\frac{p_\phi}{r} \cdot \left(-\frac{1}{M} - \frac{1}{M} + \frac{1}{M} - \frac{m}{M^2} + \frac{1}{M} + \frac{m}{M^2} \right.$$

$$\left. - \frac{1}{M} - 2\frac{m}{M^2} \right) \Big]$$

$$+ \frac{1}{2}(M+m) g R\, \theta^2 + \frac{1}{2} m g r\, \phi^2$$

$$= \frac{1}{2M}\left(\frac{p_\theta}{R}\right)^2 + \left(\frac{1}{2m} + \frac{1}{2M}\right)\left(\frac{p_\phi}{r}\right)^2 - \frac{1}{M}\frac{p_\theta}{R}\frac{p_\phi}{r}$$

$$+ \frac{1}{2}(M+m) g R\, \theta^2 + \frac{1}{2} m g r\, \phi^2$$

$$= \frac{1}{2M}\left(\frac{p_\theta}{R} - \frac{p_\phi}{r}\right)^2 + \frac{1}{2m}\left(\frac{p_\phi}{r}\right)^2 + \frac{1}{2}(M+m) g R\, \theta^2$$

$$+ \frac{1}{2} m g r\, \phi^2$$

$$\rightarrow \frac{1}{2MR^2}\left((p_\theta - p_\phi)^2 + p_\phi^2\right) \sim \frac{1}{2} M g R \left(\theta^2 + 2\phi^2\right)$$

which is manifestly positive definite.

Hamilton's equations are

$$\dot{q}^i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial q^i}$$

so

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{1}{MR}\left(\frac{p_\theta}{R} - \frac{p_\phi}{r}\right)$$

$$\dot{\phi} = \frac{\partial H}{\partial p_\phi} = -\frac{1}{Mr}\left(\frac{p_\theta}{R} - \frac{p_\phi}{r}\right) + \frac{1}{mr^2} p_\phi$$

$$= -\frac{1}{MrR} p_\theta + \frac{1}{mr^2}\left(1 + \frac{m}{M}\right) p_\phi$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -(M+m)gR\theta$$

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = -mgr\phi$$

Since the Hamiltonian isn't explicit time-dependent

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = 0$$

i.e. it is conserved.

These are just four coupled first-order ODEs, so can try an Ansatz and then get an eigenvalue problem.

2. Consider the Hamiltonian

$$H(q,p) = \frac{p_1^2 + p_2^2}{2m} + \frac{k}{2}(q_1 - q_2)^2$$

a) G generates symmetry when H left invariant.

For $G = p_1 + p_2$, change in H is

$$\delta H = \{H, G\} = \frac{\partial H}{\partial q^i}\frac{\partial G}{\partial p_i} - \frac{\partial G}{\partial q^i}\frac{\partial H}{\partial p_i}$$

$$= \frac{\partial H}{\partial q^1}\frac{\partial G}{\partial p_1} + \frac{\partial H}{\partial q^2}\frac{\partial G}{\partial p_2} - \frac{\partial G}{\partial q^1}\frac{\partial H}{\partial p_1} - \frac{\partial G}{\partial q^2}\frac{\partial H}{\partial p_2}$$

$$= k(q_1 - q_2) + -k(q_1 - q_2) = \underline{\underline{0}}$$

$\longrightarrow$ G generates symmetry.

b) since

$$\{H, G\} = 0$$

then

$$\frac{dG}{dt} = \frac{\partial G}{\partial t} + \{G, H\} = 0$$

→ G is conserved.

Explicitly, using Hamilton's equations

$$\frac{dG}{dt} = \dot{p}_1 + \dot{p}_2 = -\left(\frac{\partial H}{\partial q^1} + \frac{\partial H}{\partial q^2}\right)$$

$$= -\left(k(q_1 - q_2) + -k(q_1 - q_2)\right) = 0$$

→ total momentum

As transformation

$$\delta p^i = \frac{\partial G}{\partial q^i} \cdot \delta\lambda = 0$$

$$\delta q^i = -\frac{\partial G}{\partial p^i} \cdot \delta\lambda = -\delta\lambda$$

→ translations

c) H is not explicitly dependent on time, so

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \{H, H\} = 0$$

→ H is conserved.