ACP: RF 7

Dynamical Systems

Dynamical system with n dynamical degrees of freedom

$$\frac{dx^i}{dt} = F^i(x) \quad , \quad i \in \{1, \dots, n\}$$

→ Flow equations for trajectories: Good way to demonstrate dynamics!

Example: Hamiltonian evolution describes dynamical systems with even n.

Note that a generic higher-derivative system

$$\frac{d^N x^i}{dt^N} = F^i\left(x, x^{(1)}, \dots, x^{(N-1)}\right)$$

can be reduced to first-derivatives using

$$x^i_m = \frac{d^m x^i}{dt^m}$$

→ system of $N \times n$ first-order equations!

Critical points or equilibria $x^i = x^i_c$ when

$$F^i(x = x_c) = 0$$

Taylor expanding $x = x_c + \delta x$,

$$F^i(x) = F^i(x = x_c) + \left.\frac{\partial F^i}{\partial x^j}\right|_{x = x_c} \cdot \delta x^j + \dots$$

$$= \partial_j F^i \Big|_{x = x_c} \delta x^j + \dots$$

gives

$$\delta \dot{x}^i = \left.\frac{\partial \dot{x}^i}{\partial x^j}\right|_{x=x_c} \cdot \delta x^j + \dots$$

coupled 1st-order ODEs, so try

$$\delta x^i = A^i e^{\lambda t}$$

$$\rightarrow \quad \left.\frac{\partial \dot{x}^i}{\partial x^j}\right|_{x=x_c} A^j = \lambda A^i$$

i.e. eigenvalue problem:

- $\mathrm{Re}\{\lambda\}$: radial direction from x_c → stability?
- $\mathrm{Im}\{\lambda\}$: angular direction about x_c

Integrable systems

Dynamical systems are integrable when there are enough conserved quantities so that the equations can be fully solved.

→ look at Hamiltonian systems

Definition (Liouville integrability): integrable system is $2n$-dim. "phase space" with n independent functions f_i in involution i.e.

$$\{f_i, f_j\} = 0 \qquad \forall i,j$$

and

$$H = H(f_i)$$

theorem (Arnol'd - Liouville): for integrable system (H, f_i), consider level surface of f

$$M_f = \{ (q,p) \in M \mid f_i = c_i \}, \quad c_i = \text{const.}$$

If M is compact and connected, then diffeomorphic to

$$M_f \cong T^n = S^1 \times \dots \times S^1.$$

Local coordinates (ϕ_k, I_k), $k \in \{1, \dots, n\}$ with angular coordinates

$$\phi_k \sim \phi_k + 2\pi$$

on M_f and I_k ← phase space volume!

$$I_k = \frac{1}{2\pi} \oint_{\text{cycle}} p_j \, dq^j$$

are conserved.

→ Equations of motion:

$$\dot{I}_k = 0, \qquad \dot{\phi}_k = \omega_k(I_i)$$

which are trivially solved.

ACP: Problem Sheet 9

Dynamical systems & FT

1. Consider the second-order system of coupled ODEs:

$$\dot{x}^i = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} x - x^2 - \frac{1}{2}xy \\ y - y^2 \end{pmatrix} = F^i$$

a) Hence, critical points x_c^i when

$$F^i(x^i = x_c^i) = 0$$

First consider

$$0 = y - y^2 = y(1-y) \longrightarrow y = 0, \; y = 1$$

For $y = 0$:

$$0 = x - x^2 = x(1-x)$$

$$\longrightarrow \underline{x_{c_1} = (0,0)}, \quad \underline{x_{c_2} = (1,0)}$$

For $y = 1$,

$$0 = x - x^2 - \frac{1}{2}x = x\left(\frac{1}{2} - x\right)$$

$$\longrightarrow \underline{x_{c_3} = (0,1)}, \quad \underline{x_{c_4} = \left(\frac{1}{2}, 1\right)}$$

b) For small perturbations δx^i about the critical points x_c^i, the linearised equations of motion

$$\delta\dot{x}^i = \left.\frac{\partial F^i}{\partial x^j}\right|_{x^i = x_c^i} \delta x^j$$

$\longrightarrow$ eigenvalue problem!

In particular

$$M = \begin{pmatrix} \partial F^1/\partial x & \partial F^1/\partial y \\ \partial F^2/\partial x & \partial F^2/\partial y \end{pmatrix}\Big|_{x=x_c} = \begin{pmatrix} -1 - 3x - \frac{3}{2}y & -\frac{3}{2}x \\ 0 & -1 - 2y \end{pmatrix}\Big|_{x^i = x^i_c}$$

$$= \begin{pmatrix} -1 - 3x_c - \frac{3}{2}y_c & -\frac{3}{2}x_c \\ 0 & -1 - 2y_c \end{pmatrix}$$

Check the different critical points

$\underline{x_{c_1} = (0,0)}$:

$$M_0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

so, trivially

$$\lambda_1 = \lambda_2 = -1, \quad v_1 = (1,0), \quad v_2 = (0,1)$$

$\rightarrow$ unstable along v_1 and v_2

$\underline{x_{c_2} = (-1,0)}$:

$$M_1 = \begin{pmatrix} 2 & -\frac{3}{2} \\ 0 & 1 \end{pmatrix}$$

since

$$\mathrm{tr}(M_1) = 3, \quad \det(M_1) = 2$$

then

$$\lambda_1 = 2, \quad \lambda_2 = 1$$

For the eigenvectors

$$M\underline{v} = \begin{pmatrix} -1 & -\frac{\pi}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -v_1 - \frac{\pi}{2} v_2 \\ v_2 \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

so

$$\underline{v}_1 = \left(-\frac{\pi}{4}, 1\right), \quad \underline{v}_2 = (1, 0)$$

$\longrightarrow$ unstable along $\underline{v}_1$ and stable along $\underline{v}_2$

$\underline{x_{c3} = (0, 1)}$:

$$M_3 = \begin{pmatrix} \frac{\pi}{2} & 0 \\ 0 & -1 \end{pmatrix}$$

so, trivially

$$\lambda_1 = \frac{\pi}{2}, \quad \lambda_2 = -1, \quad \underline{v}_1 = [1, 0], \quad \underline{v}_2 = (0, 1)$$

$\longrightarrow$ unstable along $\underline{v}_1$ and stable along $\underline{v}_2$

$\underline{x_{c4} = \left(\frac{\pi}{2}, 1\right)}$:

$$M_4 = \begin{pmatrix} -\frac{\pi}{2} & -\frac{\pi}{4} \\ 0 & -1 \end{pmatrix}$$

for which

$$\tau(M_4) = -\frac{3}{2} \qquad \det(M_4) = \frac{\pi}{2}$$

so

$$\lambda_1 = -\frac{1}{2}, \quad \lambda_2 = -1$$

and for the eigenvectors:

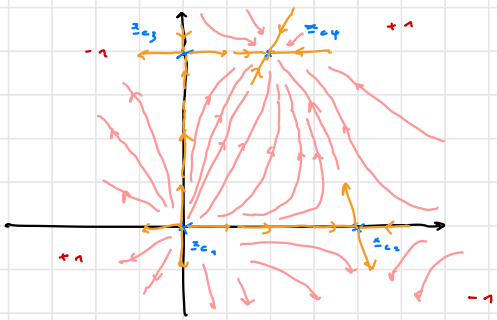
$$M_+ \underline{v} = \begin{pmatrix} -\frac{3}{2} & -\frac{7}{4} \\ 1 & -1 \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = \begin{pmatrix} -\frac{3}{2} v^1 - \frac{7}{4} v^2 \\ -v^2 \end{pmatrix}$$

from union we know from

$$\underline{v}_1 = (1, 0) \quad , \quad \underline{v}_2 = \left(\frac{7}{2}, 1\right)$$

$\longrightarrow$ stable along $\underline{v}_1$ and $\underline{v}_2$

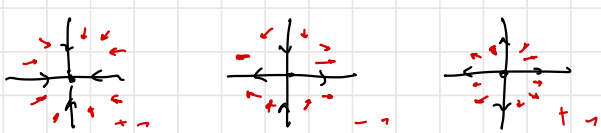
c) can now sketch phase portrait :



can't extrapolate from the linear regime

note : this is topologically necessary !

can compute of the index $\text{ind}(x_c)$ of zero of vector field as degree (winding number) of $\phi : \partial D \to S^1$, $v \longmapsto v/|v|$



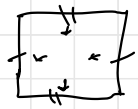
or just draw it !

The Poincaré - Hopf theorem states : for a smooth vector field,

$$2 - 2g(M) = \chi(M) = \sum_x \text{ind}(x)$$

In our case

$$\chi(M) = 1 - 1 + 1 - 1 = 0$$



→ system lives on a torus!

note: the most famous Hairy-ball theorem is a corollary of the Poincaré-Hopf theorem!

(integrability ff)

2. harmonic oscillator with Hamiltonian

$$H(q,p) = \frac{p^2}{2m} + \frac{k}{2} q^2$$

a) Hamiltonian not explicitly time-dependent, i.e.

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = 0$$

→ conserved.

b) Dynamical equations are

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} \partial H/\partial p \\ -\partial H/\partial q \end{pmatrix} = \begin{pmatrix} p/m \\ -kq \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{m} \\ -k & 0 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}$$

Slope:

$$\frac{dp}{dq} = \frac{\dot{p}}{\dot{q}} = \frac{-kq}{p/m} = -km\,\frac{q}{p}$$

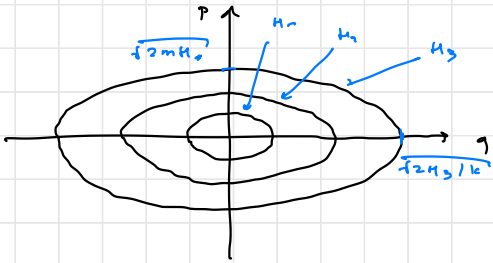
$$p^2 + km\,q^2 = \text{const}$$

→ ellipses with constant H. semi-major axes:

$$H(r_1, 0) = \frac{k}{2} \rightarrow r_1 = \sqrt{\frac{2H}{k}}$$

$$H(0, r_2) = \frac{1}{2m} p^2 \rightarrow r_2 = \sqrt{2mH}$$

Phase portrait



c) change to action - angle variables $(q, p) \mapsto (\phi, I)$

$$q = \sqrt{2Ia}\sin\phi \qquad p = \sqrt{2Ib}\cos\phi$$

Change of variables iff canonical poisson bracket satisfied, i.e.

$$1 = \{q, p\}_{\phi, I} = \frac{\partial q}{\partial \phi}\frac{\partial p}{\partial I} - \frac{\partial q}{\partial I}\frac{\partial p}{\partial \phi}$$

$$= \sqrt{2Ia}\cos\phi \cdot \sqrt{2Ib} \cdot \frac{1}{2\sqrt{I}}\cos\phi - \frac{\sqrt{2a}}{2\sqrt{I}}\sin\phi \cdot -\sqrt{2Ib}\sin\phi$$

$$= \sqrt{ab}\,(\cos^2\phi + \sin^2\phi) = \sqrt{ab}$$

$$\longrightarrow \quad b = a^{-1}$$

d)

$$H = \frac{p^2}{2m} + \frac{k}{2}q^2$$

$$= \frac{1}{2m} \cdot 2Ib\cos^2\phi + \frac{k}{2} \cdot 2Ia\sin^2\phi$$

$$= \frac{I}{am}\cos^2\phi + Ika\sin^2\phi$$

This is independent of ϕ when

$$\frac{1}{am} = ka$$

$$a^2 = \frac{1}{4m} \longrightarrow a = \frac{1}{\sqrt{10m}}$$

Then

$$H = Ika\left(\sin^2\phi + \cos^2\phi\right) = Ik \cdot \frac{1}{\sqrt{4m}}$$

$$= I \cdot \sqrt{\frac{k}{m}} = I\omega$$

for

$$\omega^2 = \frac{k}{m}$$

e) Hamilton's equations are

$$\dot{I} = -\frac{\partial H}{\partial \phi} = 0 \longrightarrow I = \frac{H}{\omega} = \text{const}.$$

$$\dot{\phi} = \frac{\partial H}{\partial I} = \omega$$

f) Area of ellipse is

$$A = \pi a_1 a_2 = \pi \cdot \sqrt{\frac{2H}{k}} \cdot \sqrt{2mH} = 2\pi \sqrt{\frac{m}{k}}\, H$$

$$= \underline{2\pi I}$$

→ as expected for $n = 1$!