ACP: RF 8

Symmetries of Minkowski space

Symmetries of Minkowski space are described by the Poincaré group:

$$x^\mu \to \Lambda^\mu{}_\nu x^\nu + a^\mu$$

translations!

→ 10 elements!

Subgroup without translations is the Lorentz group $O(1,3)$

$$x^\mu \to \Lambda^\mu{}_\nu x^\nu$$

where Λ is defined to preserve spacetime intervals

$$\Lambda^\mu{}_\alpha \eta_{\mu\nu} \Lambda^\nu{}_\beta = \eta_{\alpha\beta}$$

Discrete subgroups:

- orthochronous $O^+(1,3)$: no time reversal,

$$\Lambda^0{}_0 \geq 1$$

- proper $SO(1,3)$: no reflections

$$\det(\Lambda) = 1$$

Natural to consider

$$SO^+(1,3) \quad \text{or} \quad ISO^+(1,3)$$

Spacetime Tensors

Can be understood as multi-linear maps on tangent spaces of manifolds

→ physically more intuitive: object in representations of $SO(1,3)$, i.e. how the Lorentz group acts on?

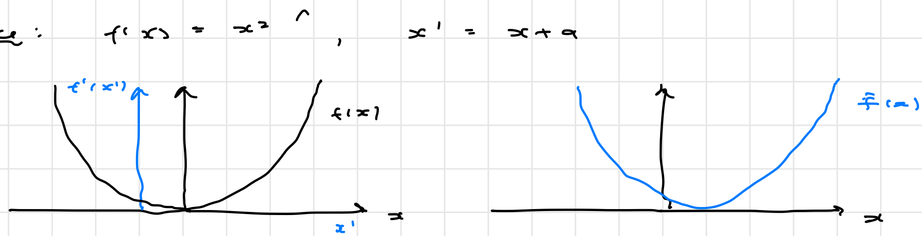
Scalar:

$$\phi(x) \mapsto \phi'(x') = \phi(x')$$ passive

$$\phi(x) \mapsto \phi'(x) = \phi(\Lambda^{-1} x)$$ active

→ stick to active

Example: $f(x) = x^2$, $x' = x + a$



passive: just relabel coordinates

active: change coordinates but insist on old coordinates

vector: Associate to vector field / arrow

so need to change orientation too

$$V^\mu(x) = \Lambda^\mu{}_\nu V^\nu(\Lambda^{-1}x)$$

now <u>general tensor</u> of rank (p,q)

$$T^{\mu_1 \dots \mu_p}{}_{\nu_1 \dots \nu_q}(x)$$

$$\longmapsto \Lambda^{\mu_1}{}_{\alpha_1} \dots \Lambda^{\mu_p}{}_{\alpha_p} \Lambda_{\nu_1}{}^{\beta_1} \dots \Lambda_{\nu_q}{}^{\beta_q} T^{\alpha_1 \dots \alpha_p}{}_{\beta_1 \dots \beta_q}(\Lambda^{-1}x)$$

ACP: Problem sheet 8

1. Start trajectory at

$$x^\mu = (t, x, y, z) = 0$$

Travel with spatial velocity

$$v^i = (v^x, v^y, v^z)$$

Travelling along the $x=y$ diagonal corresponds to $v^x = v^y$ and $v^z = 0$. With affine parameterisation

$$x^\mu = (t, v^x t, v^y t, v^z t)$$

Then, spacetime interval is

$$\begin{aligned} s^2 &= \eta_{\mu\nu} x^\mu x^\nu \\ &= -t^2 + (v^x t)^2 + (v^y t)^2 + (v^z t)^2 \\ &= (-1 + |\underline{v}|^2) t^2 \end{aligned}$$

$$\begin{cases} < 0 & \text{if } v < 1 \quad (\text{slower than light}) \\ = 0 & \text{if } v = 1 \quad (\text{speed of light}) \\ > 0 & \text{if } v > 1 \quad (\text{faster than light}) \end{cases}$$

2. Boost along the x-direction realised by

$$\Lambda = \begin{pmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \left(\begin{array}{c|c} \Lambda_2 & 0_2 \\ \hline 0_2 & I_2 \end{array}\right)$$

where

$$\gamma = \frac{1}{\sqrt{1-v^2}}$$

Everything is block-diagonal, so focus on x-y dimension:

$$\Lambda^\mu{}_\alpha \eta_{\mu\nu} \Lambda^\nu{}_\beta = (\Lambda^T \eta \Lambda)_{\alpha\beta} = \left[\begin{pmatrix} \Lambda_2 & 0 \\ 0 & I_2 \end{pmatrix}^T \begin{pmatrix} \eta_2 & 0 \\ 0 & I_2 \end{pmatrix} \begin{pmatrix} \Lambda_2 & 0 \\ 0 & I_2 \end{pmatrix}\right]_{\alpha\beta}$$

otherwise abuse of notation

$$= \left[\begin{pmatrix} \Lambda_2^T & 0 \\ 0 & I_2^T \end{pmatrix} \begin{pmatrix} \eta_2 \Lambda_2 & 0 \\ 0 & I_2 \end{pmatrix}\right]_{\alpha\beta} = \begin{pmatrix} \Lambda_2^T \eta_2 \Lambda_2 & 0 \\ 0 & I_2 \end{pmatrix}_{\alpha\beta}$$

In particular

$$\Lambda_2^T \eta_2 \Lambda_2 = \begin{pmatrix} \gamma & -v\gamma \\ -v\gamma & \gamma \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & -v\gamma \\ -v\gamma & \gamma \end{pmatrix}$$

$$= \begin{pmatrix} \gamma & -v\gamma \\ -v\gamma & \gamma \end{pmatrix} \begin{pmatrix} -\gamma & v\gamma \\ -v\gamma & \gamma \end{pmatrix} = \begin{pmatrix} -\gamma^2 + v^2\gamma^2 & v\gamma^2 - v\gamma^2 \\ v\gamma^2 - v\gamma^2 & -v^2\gamma^2 + \gamma^2 \end{pmatrix}$$

$$= \gamma^2(1-v^2) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

so that

$$\Lambda^\mu{}_\alpha \eta_{\mu\nu} \Lambda^\nu{}_\beta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{\alpha\beta} = \eta_{\alpha\beta}$$

as required.

3. Rotations about x-axis are realised by

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix} = \left(\begin{array}{c|c} I_2 & 0_2 \\ \hline 0_2 & R_2 \end{array}\right)$$

Then

$$R^\mu{}_\alpha \eta_{\mu\nu} R^\nu{}_\beta = \left[\begin{pmatrix} I_2 & 0 \\ 0 & R_2 \end{pmatrix}^T \begin{pmatrix} \eta_2 & 0 \\ 0 & I_2 \end{pmatrix} \begin{pmatrix} I_2 & 0 \\ 0 & R_2 \end{pmatrix} \right]_{\alpha\beta}$$

$$= \left[\begin{pmatrix} I_2 & 0 \\ 0 & R_2^T \end{pmatrix} \begin{pmatrix} \eta_2 I_2 & 0 \\ 0 & I_2 R_2 \end{pmatrix} \right]_{\alpha\beta} = \begin{pmatrix} I_2 \eta_2 I_2 & 0 \\ 0 & R_2^T I_2 R_2 \end{pmatrix}_{\alpha\beta}$$

where

$$I_2 \eta_2 I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$R_2^T I_2 R_2 = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \cos^2\theta + \sin^2\theta \end{pmatrix}$$

$$= I_2$$

so

$$R^\mu{}_\alpha \eta_{\mu\nu} R^\nu{}_\beta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{\alpha\beta} = \eta_{\alpha\beta}$$

as required.

Note: This is the defining relation for Λ. Can in fact derive everything else!

Take infinitesimal transformation

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$$

then

$$\Lambda^\mu{}_\alpha \, \eta_{\mu\nu} \, \Lambda^\nu{}_\beta$$

$$= (\delta^\mu{}_\alpha + \omega^\mu{}_\alpha) \, \eta_{\mu\nu} \, (\delta^\nu{}_\beta + \omega^\nu{}_\beta)$$

$$= (\delta^\mu{}_\alpha \delta^\nu{}_\beta + \delta^\mu{}_\alpha \omega^\nu{}_\beta + \omega^\mu{}_\alpha \delta^\nu{}_\beta + O(\omega^2)) \, \eta_{\mu\nu}$$

$$= \eta_{\alpha\beta} + \omega_{\alpha\beta} + \omega_{\beta\alpha} + O(\omega^2)$$

$$= \eta_{\alpha\beta}$$

so

$$\omega_{\alpha\beta} = -\omega_{\beta\alpha}$$

or

3 boosts

$$\omega = \begin{pmatrix} 0 & \omega_{01} & \omega_{02} & \omega_{03} \\ -\omega_{01} & 0 & \omega_{12} & \omega_{13} \\ -\omega_{02} & -\omega_{12} & 0 & \omega_{23} \\ -\omega_{03} & -\omega_{13} & -\omega_{23} & 0 \end{pmatrix}$$

3 rotations!

can implement as

$$\omega_{i0} = -\hat{n}_i \cdot \eta \qquad \omega_{ij} = -\varepsilon_{ijk} \hat{n}_k \theta$$