ACP: RF 9

Spin-1 gauge field described by field

$$A^\mu(t, x)$$

and Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

where

$$F_{\mu\nu} = 2\, \partial_{[\mu} A_{\nu]}$$

In terms of E and B

$$F_{0i} = -E_i \quad , \quad F_{ij} = \epsilon_{ijk} B^k$$

The Lorentz force law is

$$\frac{dp^\mu}{d\tau} = q F^\mu{}_\nu u^\nu$$

ACP: problem sheet 9

1.

a) Inertial frame with

$$x^\mu = (t, x, y, z)$$

particle of mass m and charge q, velocity along the x-direction, in $\underline{E} = 0$

$$F_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & B_z & -B_y \\ 0 & \cdot & 0 & B_x \\ 0 & \cdot & \cdot & 0 \end{pmatrix}_{\mu\nu}$$

Then

$$F_{\mu\nu} F^{\mu\nu} = 2 F_{0i} F^{0i} + F_{ij} F^{ij}$$

$$= -2 E_i E^i + \varepsilon_{ijm} B^m \cdot \varepsilon^{ijn} B_n$$

$$= -2 \cancel{|\underline{E}|^2} + 2 \cdot \delta_m{}^n \cdot B^m B_n = 2 |\underline{B}|^2$$

Lorentz scalar, so invariant under Lorentz transformation.

b) switch to particle rest frame

$$x'^\mu = (t', x', y', z')$$

related to x^μ by boost along z by $\underline{-v}$

$$\Lambda = \begin{pmatrix} \gamma & 0 & 0 & v\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ v\gamma & 0 & 0 & \gamma \end{pmatrix}$$

so

$$F'_{\mu\nu} = \Lambda_\mu{}^\alpha F_{\alpha\beta} \Lambda_\nu{}^\beta \Rightarrow (\Lambda F \Lambda^T)_{\mu\nu}$$

where

$$\Lambda F \Lambda^T = \begin{pmatrix} \gamma & 0 & 0 & v\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ v\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & B_z & -B_y \\ 0 & -B_z & 0 & B_x \\ 0 & B_y & -B_x & 0 \end{pmatrix} \begin{pmatrix} \gamma & 0 & 0 & v\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ v\gamma & 0 & 0 & \gamma \end{pmatrix}$$

$$= \begin{pmatrix} \gamma & 0 & 0 & v\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ v\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ -v\gamma B_y & 0 & B_z & -\gamma B_y \\ v\gamma B_x & -B_z & 0 & \gamma B_x \\ 0 & B_y & -B_x & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & v\gamma B_y & -v\gamma B_x & 0 \\ \cdot & 0 & B_z & -\gamma B_y \\ \cdot & \cdot & 0 & \gamma B_x \\ \cdot & \cdot & \cdot & 0 \end{pmatrix} = \begin{pmatrix} 0 & -E_x' & -E_y' & -E_z' \\ \cdot & 0 & B_z' & -B_y' \\ \cdot & \cdot & 0 & B_x' \\ \cdot & \cdot & \cdot & 0 \end{pmatrix}$$

so that

$$\underline{E}' = (-v\gamma B_y, \quad v\gamma B_x, \quad 0)$$

$$\underline{B}' = (\gamma B_x, \quad \gamma B_y, \quad B_z)$$

check

$$F'_{\mu\nu} F'^{\mu\nu} = 2 F'_{0i} F'^{0i} + F'_{ij} F'^{ij}$$

$$= -2 E_i' E'^i + \epsilon_{ijk} B'^k \epsilon^{ij}{}_\ell B'^\ell$$

$$= 2 \left(-|\underline{E}'|^2 + |\underline{B}'|^2 \right)$$

$$= 2 \left(-v^2\gamma^2 (B_y^2 + B_x^2) + \gamma^2 (B_x^2 + B_y^2) + B_z^2 \right)$$

$$= 2 \left[(B_x^2 + B_y^2)(\gamma^2 - v^2\gamma^2) + B_z^2 \right)$$

$$= 2\left(B_x^2 + B_y^2 + B_z^2\right) = \underline{2|\underline{B}|^2}$$

$\longrightarrow$ invariant, as expected!

c) The particle's 4-momentum is given by the Lorentz force law

$$\frac{dp^\mu}{d\tau} = qF^\mu{}_\nu u^\nu$$

in the rest frame, where

$$u^\mu = (1, 0, 0, 0)$$

we find

$$\frac{dp^\mu}{d\tau} = q\begin{pmatrix} 0 & -v\gamma B_y & v\gamma B_x & 0 \\ \cdot & 0 & B_z & -\gamma B_y \\ \cdot & \cdot & 0 & \gamma B_x \\ \cdot & \cdot & \cdot & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ v\gamma B_y \\ -v\gamma B_x \\ 0 \end{pmatrix}$$

so the force due to B acting on v is induced by the electric field in the rest frame!