ACP: RF 10

Spin-1 gauge field described by

$$A^\mu(t,x)$$

Associated to field strength

$$F_{\mu\nu} = (dA)_{\mu\nu} = 2\partial_{[\mu}A_{\nu]} \doteq \partial_\mu A_\nu - \partial_\nu A_\mu$$

In terms of $\vec{E}$ and $\vec{B}$:

$$F_{0i} = \partial_0 A_i - \partial_i A_0 = (\nabla\phi)_i + \dot{A}_i$$

$$= -E_i$$

$$F_{ij} = \partial_i A_j - \partial_j A_i = -\varepsilon_{ijk} B^k$$

To see the latter,

$$\varepsilon^{ijm} F_{ij} = -\varepsilon^{ijm}\varepsilon_{ijk} B^k = -2\delta^m_k B^k = -2B^m$$

$$B^k = -\frac{1}{2}\varepsilon^{ijk} F_{ij}$$

This automatically satisfies <u>Bianchi identity</u>

$$(dF)_{\lambda\mu\nu} = 3\partial_{[\lambda}F_{\mu\nu]}$$

$$= \partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu}$$

$$= \partial_\lambda(\cancel{\partial_\mu A_\nu} - \cancel{\partial_\nu A_\mu}) + \partial_\mu(\cancel{\partial_\nu A_\lambda} - \cancel{\partial_\lambda A_\nu})$$

$$+ \partial_\nu(\cancel{\partial_\lambda A_\mu} - \cancel{\partial_\mu A_\lambda})$$

$$= 0$$

hence

$$(\star\, dF)_\alpha = \frac{1}{3!}\,\varepsilon_{\alpha\lambda\mu\nu}\,(dF)^{\lambda\mu\nu} = \frac{1}{2}\,\varepsilon_{\alpha\lambda\mu\nu}\,\partial^{[\lambda}F^{\mu\nu]}$$

$$= \frac{1}{2}\,\varepsilon_{\alpha\lambda\mu\nu}\,\partial^\lambda F^{\mu\nu} = 0$$

check

$\underline{\alpha = 0}$:

$$(\star\, dF)_0 = \frac{1}{2}\,\varepsilon_{0\lambda\mu\nu}\,\partial^\lambda F^{\mu\nu} = \frac{1}{2}\,\varepsilon_{0ijk}\,\partial^i F^{jk}$$

$$= \frac{1}{2}\,\varepsilon_{ijk}\,\partial^i\cdot\,\varepsilon^{jkm}B_m = \partial_i B^i$$

$\underline{\alpha = i}$:

$$(\star\, dF)_i = \frac{1}{2}\,\varepsilon_{i\lambda\mu\nu}\,\partial^\lambda F^{\mu\nu}$$

$$= \frac{1}{2}\,\varepsilon_{i0jk}\,\partial^0 F^{jk} + \frac{1}{2}\,\varepsilon_{ij0k}\,\partial^j F^{0k}$$

$$+ \frac{1}{2}\,\varepsilon_{ijk0}\,\partial^j F^{k0}$$

$$= -\frac{1}{2}\,\varepsilon_{ijk}\,\partial^0\left(\varepsilon^{jkm}B_m\right) + \frac{1}{2}\,\varepsilon_{ijk}\,\partial^j\left(E^k\right)$$

$$- \frac{1}{2}\,\varepsilon_{ijk}\,\partial^j\left(-E^k\right)$$

$$= -\partial^0 B_i + \varepsilon_{ijk}\,\partial^j E^k = \left(\dot{\underline{B}} + \underline{\nabla}\times\underline{E}\right)_i$$

$\longrightarrow$ Magnetic Gauss & Faraday

other Maxwell's equations are equivalent to

$$(d\star F)_{\lambda\mu\nu} = 3\,\partial_{[\lambda}(\star F)_{\mu\nu]}$$

$$= 3\,\partial_\lambda\left(\frac{1}{2!}\,\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}\right) = \frac{3}{2}\,\varepsilon_{\mu\nu\alpha\beta}\,\partial_\lambda F^{\alpha\beta} = 0$$

this implies that

$$(\star\, d\star F)_\kappa = \frac{1}{3!}\,\varepsilon_\kappa{}^{\lambda\mu\nu}\cdot(d\star F)_{\lambda\mu\nu}$$

$$= \frac{1}{2^2} \, \varepsilon_{\kappa\lambda\mu\nu} \cdot \varepsilon^{\mu\nu\alpha\beta} \, \partial^\lambda F_{\alpha\beta}$$

$$= \frac{1}{2^2} \cdot -2 \cdot 2 \, \delta^\alpha_{[\kappa} \delta^\beta_{\lambda]} \, \partial^\lambda F_{\alpha\beta} = - \partial^\lambda F_{\kappa\lambda} = \partial^\lambda F_{\lambda\kappa} = 0$$

check

$\underline{\kappa = 0}$: $(\star d \star F)_0 = \partial_\mu F^\mu{}_0 = \partial_i F^i{}_0 = \partial_i E^i$

$$= \underline{\nabla} \cdot \underline{E}$$

$\underline{\kappa = i}$: $(\star d \star F)_i = \partial_\mu F^\mu{}_i = \partial_0 F^0{}_i + \partial_j F^j{}_i$

$$= \partial_0 E_i + \partial_j \cdot \varepsilon^j{}_{ik} B^k$$

$$= \dot{E}_i - (\underline{\nabla} \times \underline{B})_i$$

$\longrightarrow$ electric Gauss & Ampère

This is the equation of motion for the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

check:

$$0 = \underbrace{\cancel{\frac{\partial \mathcal{L}}{\partial A_\mu}}}_{=0} - \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} \right) = \frac{1}{4} \partial_\nu \left(\frac{\partial (F_{\alpha\beta} F^{\alpha\beta})}{\partial(\partial_\nu A_\mu)} \right)$$

$$= \frac{1}{4} \partial_\nu \left(\frac{\partial F_{\alpha\beta}}{\partial(\partial_\nu A_\mu)} F^{\alpha\beta} + F_{\alpha\beta} \frac{\partial F^{\alpha\beta}}{\partial(\partial_\nu A_\mu)} \right)$$

$$= -\frac{1}{2} \partial_\nu \left(\frac{\partial F_{\alpha\beta}}{\partial(\partial_\nu A_\mu)} F^{\alpha\beta} \right)$$

$$= -\frac{1}{2} \partial_\nu \left((\delta^\nu_\alpha \delta^\mu_\beta - \delta^\nu_\beta \delta^\mu_\alpha) F^{\alpha\beta} \right)$$

$$= -\frac{1}{2} \partial_\nu (F^{\nu\mu} - F^{\mu\nu}) = \underline{-\partial_\nu F^{\nu\mu}}$$

as required.

Action is invariant under gauge transformations

$$A^\mu \to A^\mu + \partial^\mu \alpha$$

since

$$F^{\mu\nu} = 2\,\partial^{[\mu} A^{\nu]} \to F^{\mu\nu} + 2\,\partial^{[\mu}\partial^{\nu]}\alpha = F^{\mu\nu}$$

Coupling to matter should respect this: take scalar

$$\mathcal{L} \supset \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2$$

can't have

$$\mathcal{L} \not\ni A^\mu \phi\, \partial_\mu \phi \to A^\mu \phi\, \partial_\mu \phi + \partial^\mu \alpha\, \phi\, \partial_\mu \phi$$

$\to$ not gauge invariant!

use complex field (doublet) to compensate

$$\phi = \phi_1 + i\phi_2 \qquad \phi^* = \phi_1 - i\phi_2$$

then, simultaneously rotate

$$\phi \to e^{-i\alpha(x)}\phi$$

then:

$$|\phi|^2 \to (e^{-i\alpha}\phi)(e^{i\alpha}\phi) = |\phi|^2$$

but

$$\partial_\mu \phi \to \partial_\mu(e^{-i\alpha}\phi) = e^{-i\alpha}(\partial_\mu\phi - i\,\partial_\mu\alpha \cdot \phi)$$

which makes $|\partial_\mu \phi|^2$ not gauge invariant.

instead consider the (gauge) $\underbrace{\text{covariant derivative}}$

$$D_\mu \phi = (\partial_\mu + i A_\mu)\phi$$

$$\to e^{-i\alpha}\left[\partial_\mu \phi - i\cancel{\partial_\mu \alpha}\,\phi + i(A_\mu + \cancel{\partial_\mu \alpha})\phi\right]$$

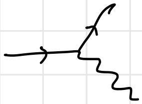
$$= e^{-i\alpha} D_\mu \phi$$

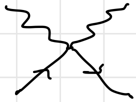
so $|D_\mu \phi|^2$ is gauge inv.

$\to$ scalar QED Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + |D_\mu \phi|^2 - m^2|\phi|^2$$

possible Feynman diagrams in QED / QCD:





$\uparrow$

not in QED

and in QCD also: