AQFT : Problem Sheet 1

1.

a) split expression

$$F[\phi] = F_1 \, e^{F_2 + F_3}$$

where

$$F_1[\phi] = \int d^D x \int d^D y \quad \phi(x) \, K(x, y) \, \phi(y)^2$$

$$F_2[\phi] = i \int d^D x \quad \phi(x) \, J(x)$$

symmetric in $\{y, z, w\}$

$$F_3[\phi] = \int d^D x \int d^D y \int d^D z \int d^D w \quad A(x, y, z, w)$$

$$\times \left(\partial \phi(x) \right)^2 \phi(y) \, \phi(z) \, \phi(w)$$

then

$$\frac{\delta F_1[\phi]}{\delta \phi(u)} = \int d^D y \quad K(u, y) \, \phi(y)^2$$

$$+ \int d^D x \quad \phi(x) \, K(x, u) \cdot 2\phi(u)$$

$$= \int d^D x \quad \phi(x) \left(K(u, x) \, \phi(x) + 2 \, K(x, u) \, \phi(u) \right)$$

$$\frac{\delta F_2[\phi]}{\delta \phi(u)} = i \, J(u)$$

For F_3, first note that

$$\delta F_3[\phi] = \int d^D x \int d^D y \int d^D z \int d^D w \quad A(x, y, z, w)$$

$$\cdot \, 2\partial\phi(x) \, \partial \delta\phi(x) \, \phi(y) \, \phi(z) \, \phi(w)$$

$$= - \int d^D x \int d^D y \int d^D z \int d^D w \quad \partial^x_\mu \left[A(x,y,z,w)\, \partial^\mu \phi(x) \right]$$

$$\cdot \, \phi(y)\, \phi(z)\, \phi(w) \cdot \quad \delta\phi(x)$$

so that

$$\frac{\delta F_3[\phi]}{\delta\phi(u)} = -2 \int d^D y \int d^D z \int d^D \cancel{w}^{x} \cdot \partial^u_\mu \left[\partial^{u,\mu} \phi(u) \right.$$

$$\left. A(u,y,z,\cancel{w}^{x}) \right] \cdot \phi(y)\, \phi(z)\, \phi(\cancel{w}^{y})$$

$$+ \cdot \int d^D x \int d^D z \int d^D \cancel{w}^{y} \; A(x,u,\cancel{w}^{y},z) \left(\partial\phi(x) \right)^2 \phi(x)\, \phi(\cancel{w}^{y})$$

$$+ \left(\text{permutations} \right)$$

$$= \int d^D x \int d^D y \int d^D z \left[-2\, \partial^u_\mu \left(\partial^{u,\mu} \phi(u)\, A(u,y,z,x) \right) \right.$$

$$\left. \cdot \, \phi(x)\, \phi(y)\, \phi(z) + 3\, A(x,y,z,u) \left(\partial\phi(x) \right)^2 \phi(y)\, \phi(z) \right]$$

and

$$\frac{\delta F[\phi]}{\delta\phi(u)} = \frac{\delta F_1}{\delta\phi(u)} e^{F_2 + F_3} + F_1 \left(\frac{\delta F_2}{\delta\phi(u)} + \frac{\delta F_3}{\delta\phi(u)} \right) e^{F_2 + F_3}$$

$$= \frac{\delta}{\delta\phi(u)} \left(\log F_1 + F_2 + F_3 \right) \cdot F[\phi]$$

b) given

drop hats $\qquad\qquad (-+\ldots+)$

$$\phi(x) = \int \frac{d^3 k}{2E_k} \left(a_k\, e^{ik\cdot x} + a_k^+\, e^{-ik\cdot x} \right)$$

$$\pi(x) = \frac{-i}{2} \int d^3 k \left(a_k\, e^{ik\cdot x} - a_k^+\, e^{-ik\cdot x} \right)$$

then

$$\sqrt{-\partial^2 + m^2}\; \phi(x) = \int \frac{d^3 k}{2} \left(a_k\, e^{ik\cdot x} + a_k^+\, e^{-ik\cdot x} \right)$$

so after inverse Fourier transform

$$a_k^+ = \int d^3x \; e^{-ik\cdot x} \left(-i\pi(x) + \sqrt{-\nabla^2+m^2}\, \phi(x) \right)$$

Also know that

$$\langle\phi|0\rangle = A \exp\left[-\frac{1}{2} \int d^D x \; \phi(x) \sqrt{-\nabla^2+m^2}\, \phi(x) \right]$$

So that

$$\frac{\delta}{\delta\phi(x)} \langle\phi|0\rangle = \left[-\sqrt{-\nabla^2+m^2}\, \phi(x) \right] \langle\phi|0\rangle$$

In field space rep.

$$\langle\phi|a_k^+|0\rangle = \int d^3x \; e^{-ik\cdot x} \langle\phi\left(-i\pi(x) + \sqrt{-\nabla^2+m^2}\, \phi(x) \right)|0\rangle$$

$$= \int d^3x \; e^{-ik\cdot x} \left(-\frac{\delta}{\delta\phi(x)} + \sqrt{-\nabla^2+m^2}\, \phi(x) \right) \langle\phi|0\rangle$$

$$= 2 \cdot \int d^3x \; e^{-ik\cdot x} \sqrt{-\nabla^2+m^2}\, \phi(x) \langle\phi|0\rangle$$

$$= 2 \cdot \underline{\int d^3x} \; \underline{e^{-ik\cdot x}} \sqrt{-\nabla^2+m^2} \cdot \int d^3p \; \underline{e^{ip\cdot x}} \; \tilde{\phi}(p) \langle\phi|0\rangle$$

$$= 2 \cdot \int d^3x \int d^3p \; \sqrt{p^2+m^2} \cdot e^{i(p-k)\cdot x} \; \tilde{\phi}(p) \langle\phi|0\rangle$$

$$= 2 \cdot \int d^3p \; \sqrt{p^2+m^2} \cdot \delta^{(3)}(p-k) \; \tilde{\phi}(p) \langle\phi|0\rangle$$

$$= \underbrace{2 \sqrt{k^2+m^2} \; \tilde{\phi}(k) \; \langle\phi|0\rangle}$$

same as integration by parts!

similarly

$$\langle\phi|a_{k_2}^+ a_{k_1}^+|0\rangle$$

$$= \int d^3y \; e^{-ik_2\cdot y} \left(\sqrt{-\nabla_y^2+m^2}\, \phi(y) - \frac{\delta}{\delta\phi(y)} \right)$$

$$\left[\int d^3x \; e^{-ik_1\cdot x} \left(\sqrt{-\nabla_x^2+m^2}\, \phi(x) - \frac{\delta}{\delta\phi(x)} \right) \langle\phi|0\rangle \right]$$

$$= \int d^3x \int d^3y \; e^{-ik_1\cdot x - ik_2\cdot y} \left(\sqrt{-\nabla_y^2 + m^2}\,\phi(y) - \frac{\delta}{\delta\phi(y)} \right)$$

$$\times 2 \left[\sqrt{\nabla_x^2 + m^2}\,\phi(x) \langle \phi | 0 \rangle \right]$$

$$= 2 \int d^3x \int d^3y \; e^{-ik_1\cdot x - ik_2\cdot y} \Big(\underline{\sqrt{-\nabla_y^2 + m^2}\,\sqrt{-\nabla_x^2 + m^2}\,\phi(x)\,\phi(y)}$$

$$- \sqrt{\nabla_x^2 + m^2} \cdot \delta^{(3)}(x-y) + \underline{\sqrt{\nabla_x^2 + m^2} \cdot \phi(x) \sqrt{\nabla_y^2 + m^2}\,\phi(y)} \Big)$$

$$\cdot \langle \phi | 0 \rangle$$

integrated by parts?

$$= \Big(4 \int d^3x \int d^3y \; e^{-ik_1\cdot x - ik_2\cdot y} \sqrt{k_2^2 + m^2} \cdot \sqrt{k_1^2 + m^2}\,\phi(x)\,\phi(y)$$

$$- 2 \int d^3x \; e^{-i(k_1+k_2)\cdot x} \sqrt{k_1^2 + m^2} \Big) \langle \phi | 0 \rangle$$

$$= \Big(4 \cdot \sqrt{k_2^2 + m^2}\,\sqrt{k_1^2 + m^2}\;\tilde{\phi}(k_1)\,\tilde{\phi}(k_2)$$

$$- 2 \sqrt{k_1^2 + m^2}\;\delta^{(3)}(k_1 + k_2) \Big) \langle \phi | 0 \rangle$$

2.

a) Harmonic oscillator in D dimensions

$$H = \frac{1}{2m} |p|^2 + \frac{1}{2} m\omega^2 |q|^2$$

$$= \frac{1}{2m} p_i p^i + \frac{1}{2} m\omega^2 q_i q^i$$

Classical partition function $\qquad dp = \frac{dp}{2\pi}, \quad \hbar = 1$

$$Z(\beta) = \int d^D q \int d^D p \; e^{-\beta H(q,p)}$$

$$= \int d^D q \int d^D p \; \exp\left[-\frac{\beta}{2m} p_i p^i - \frac{\beta}{2} m\omega^2 q_i q^i \right]$$

$$= \prod_{i=1}^{D} \cdot \left(\int dp_i \; \exp\left[-\frac{\beta}{2m} p_i p^i \right] \right)$$

$$\cdot \left(\int dq_i \; \exp\left[-\frac{\beta}{2} m\omega^2 q_i q^i \right] \right)$$

$$= \left(\int dp\, e^{-\frac{\beta}{2m} p^2} \right)^D \left(\int dq\, e^{-\frac{\beta}{2} m\omega^2 q^2} \right)^D$$

$$= \left(\frac{1}{\cancel{2\pi}} \right)^D \left(\frac{\cancel{\pi}}{\beta\, \cancel{2}\cancel{m}} \right)^{D/2} \cdot \left(\frac{\pi}{\beta \cancel{m}\omega^2 / \cancel{2}} \right)^{D/2}$$

$$= \underline{\left(\frac{1}{\beta\omega} \right)^D}$$

classical free energy

$$F = -\frac{1}{\beta} \log Z = \underline{\frac{D}{\beta} \log (\beta \omega)}$$

b) for quantum, replace phase space integral with trace over Hilbert space

$$Z = \int d^D q\, d^D p\, e^{-\beta H(q,p)} \rightarrow \mathrm{Tr}\left(e^{-\beta \hat{H}} \right)$$

position states form complete basis, so

$$= \int d^D x\, \langle x | e^{-\beta H} | x \rangle$$

drop hats!

c) evaluate this by inserting the phase space identity

$$I = \int d^D x \int d^D p\, | x \rangle \langle x | p \rangle \langle p |$$

$$= \int d^D x \int d^D q\, \langle x | p \rangle\, | x \rangle \langle p |$$

$$= \int d^D x \int d^D p\, e^{-i x \cdot p}\, | x \rangle \langle p |$$

split $\beta = N \cdot \varepsilon$, and use I with shifted indices for matching

$$\sim | x_{i+1} \rangle \langle p_i |$$

Then, since $[H, H] = 0$

$$Z = \int d^D x\, \langle x | \underbrace{e^{-\varepsilon H} \cdots e^{-\varepsilon H}}_{N \text{ times}} | x \rangle$$

$$= \int d^D x \left(\prod_{i=0}^{N-1} \int d^D x_{i+1} \int d^D p_i \, \langle x_{i+1} | p_i \rangle \right)$$

$$\times \langle x | x_N \rangle \langle p_{N-1} | e^{-\varepsilon H} | x_{N-1} \rangle \cdots \langle p_0 | e^{-\varepsilon H} | x \rangle$$

$$= \int d^D x \cdot \left(\prod_{i=1}^{N-1} d^D x_i \right) \left(\prod_{i=0}^{N-1} \int d^D p_i \right)$$

$$\cdot \left(\prod_{i=0}^{N-1} \langle x_{i+1} | p_i \rangle \langle p_i | e^{-\varepsilon H} | x_i \rangle \right)$$

with $x_0 = x$

$$= \int d^D x \cdot \left(\prod_{i=1}^{N-1} d^D x_i \right) \left(\prod_{i=0}^{N-1} \int d^D p_i \right)$$

$$\cdot \left(\prod_{i=0}^{N-1} e^{-i p_i \cdot x_{i+1}} \cdot e^{-\varepsilon H(x_i, p_i)} \, e^{i p_i \cdot x_i} \right)$$

$$= \int d^D x \cdot \left(\prod_{i=1}^{N-1} d^D x_i \right) \left(\prod_{i=0}^{N-1} \int d^D p_i \, e^{-i p_i \cdot (x_{i+1} - x_i)} \right.$$

$$\left. e^{-\varepsilon H(x_i, p_i)} \right)$$

$$= \int d^D x \left(\prod_{i=1}^{N-1} d^D x_i \right) \left(\prod_{i=0}^{N-1} \int d^D p_i \right)$$

$$\exp \left[\sum_{i=0}^{N-1} -i\varepsilon \, p_i \cdot \frac{x_{i+1} - x_i}{\varepsilon} - \varepsilon H(x_i, p_i) \right]$$

In the continuum limit $N \to \infty$, and

$$\to \int d^D x \cdot \int_{x(0)=x}^{x(\beta)=x} \mathcal{D}x \cdot \int \mathcal{D}p \; e^{\int_0^\beta d\tau \left(-ip \cdot \frac{dx}{d\tau} - H \right)}$$

$$= \int_{x(0)=x(\beta)} \mathcal{D}x \int \mathcal{D}p \; e^{\int_0^\beta d\tau \left(-ip \cdot \frac{dx}{d\tau} - H \right)}$$

spoiler: $p\dot{q} - H$

d) Actually do the p-integrals for natural H

$$d^D p_i \exp \left[-i p_i \cdot (x_{i+1} - x_i) - \varepsilon \left(\frac{p_i p^i}{2m} + V(x_i) \right) \right]$$

$$= \int d^D p_i \exp \left[-\frac{\varepsilon}{2m} \left(p_i p^i + i \cdot \frac{2m}{\varepsilon} p_i (x_{i+1} - x_i) \right) \right.$$

$$\left. - \varepsilon V(x_i) \right]$$

$$= e^{-\varepsilon V(x_i)} \cdot \int d^D p_i \quad \exp\left\{ -\frac{\varepsilon}{2m}\left[\left(p_i + \frac{im}{\varepsilon}(x_{i+1}-x_i)\right)^2 + \frac{m^2}{\varepsilon^2}(x_{i+1}-x_i)^2 \right]\right\}$$

$$= e^{-\varepsilon V(x_i) - \frac{m}{2\varepsilon}(x_{i+1}-x_i)^2} \cdot \left(\frac{1}{2\pi} \cdot \sqrt{\frac{\pi}{\varepsilon/2m}} \right)^D$$

$$= \left(\frac{m}{2\pi\varepsilon}\right)^{D/2} \exp\left[-\varepsilon \left(\frac{1}{2}m\left(\frac{x_{i+1}-x_i}{\varepsilon}\right)^2 + V(x_i) \right)\right]$$

so that

$$Z(\beta) = \int d^D x \cdot \left(\prod_{i=1}^{N-1} d^D x_i \right)\left(\prod_{i=0}^{N-1} \left(\frac{m}{2\pi\varepsilon}\right)^{D/2} \exp\left[-\varepsilon \left(\frac{1}{2}m\left(\frac{x_{i+1}-x_i}{\varepsilon}\right)^2 + V(x_i) \right)\right]\right)$$

$$= \left(\frac{m}{2\pi\varepsilon}\right)^{ND/2} \int d^D x \left(\prod_{i=1}^{N-1} d^D x_i \right) \cdot \exp\left[-\varepsilon \sum_{i=0}^{N-1} \left(\frac{1}{2}m\left(\frac{x_{i+1}-x_i}{\varepsilon}\right)^2 + V(x_i) \right)\right]$$

In the continuum limit, $N \to \infty$, and

$$\to \mathcal{N} \int d^D x \cdot \int_{x(0)=x}^{x(\beta)=x} \mathcal{D}x \; e^{-\int_0^\beta d\tau \, L_E}$$

$$= \mathcal{N} \int_{x(0)=x(\beta)} \mathcal{D}x \; e^{-\int_0^\beta d\tau \, L_E}$$

where

$$\mathcal{N} = \lim_{N\to\infty} \left(\frac{m}{2\pi\varepsilon}\right)^{ND/2} \to \infty$$

and

$$L_E = \frac{1}{2}m\left|\frac{dx}{d\tau}\right|^2 + V(x_i)$$

e) Discrete Fourier transform

$$x_j = \frac{1}{\sqrt{N}} \sum_{k=-M}^{M} y_k \, e^{i\frac{2\pi kj}{N}}$$

with $M = \dfrac{N-1}{2}$. Then

$$x_{j+N} = \frac{1}{\sqrt{N}} \sum_{k=-M}^{M} y_k \, e^{i\frac{2\pi k}{N}(j+N)}$$

$$= \frac{1}{\sqrt{N}} \cdot \sum_{k=-M}^{M} y_k \, e^{i\frac{2\pi k j}{N} + i 2\pi k}$$

Since $k \in \mathbb{Z}$,

$$= \frac{1}{\sqrt{N}} \sum_{k=-M}^{M} y_k \, e^{i\frac{2\pi k j}{N}} = x_j$$

f) use the fact that

$$\sum_{j=0}^{N-1} e^{i2\pi j (k-k')/N} = \frac{1 - e^{i2\pi j (k-k')}}{1 - e^{i2\pi j (k-k')/N}}$$

$$= \frac{e^{i\pi j (k-k')}}{e^{i\pi j (k-k')/N}} \cdot \frac{e^{i\pi j (k'-k)} - e^{-i\pi j (k'-k)}}{e^{i\pi j (k'-k)/N} - e^{-i\pi j (k'-k)/N}}$$

$$= e^{i\pi j (k-k') \cdot \frac{N-1}{N}} \cdot \frac{\sin(\pi j (k-k'))}{\sin(\pi j (k-k')/N)}$$

for $k - k' \neq 0$, this vanishes since $j \in \mathbb{Z}$, and for $k - k' = 0$ we use L'Hospital, so

$$= \underline{N \cdot \delta_{k,k'}}$$

Look at RHS

$$\sum_{j=0}^{N-1} x_j \, e^{-i2\pi j k/N}$$

$$= \sum_{j=0}^{N-1} \left(\frac{1}{\sqrt{N}} \cdot \sum_{k'=-M}^{M} y_{k'} \, e^{i2\pi j k'/N} \right) e^{-i2\pi j k/N}$$

$$= \frac{1}{\sqrt{N}} \cdot \sum_{k'=-M}^{k=M} y_{k'} \cdot \left(\sum_{j=0}^{N-1} e^{i2\pi j (k'-k)/N} \right)$$

$$= \frac{1}{\sqrt{N}} \cdot \sum_{k'=-M}^{k=M} y_{k'} \cdot N \, \delta_{k,k'} = \sqrt{N} \, y_k$$

so

$$y_k = \frac{1}{\sqrt{N}} \cdot \sum_{j=0}^{N-1} x_j \, e^{-i 2\pi k j/N}$$

Then, we see that

$$y_k^* = \left(\frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} x_j \, e^{-i 2\pi j k/N} \right)^*$$

$$= \frac{1}{\sqrt{N}} \cdot \sum_{j=0}^{N-1} x_j^* \, e^{i 2\pi j k/N}$$

$$= \frac{1}{\sqrt{N}} \cdot \sum_{j=0}^{N-1} x_j \, e^{-i 2\pi (-j) k/N} = y_{-k}$$

g) Discretised Eucl. action

$$S_E^{dis} = \beta \sum_{j=0}^{N-1} \left(\frac{Nm}{2\beta^2} \underbrace{(x_{j+1} - x_j)^2}_{1} + \frac{1}{N} \cdot \frac{1}{2} m\omega^2 \underbrace{(x_j)^2}_{2} \right)$$

consider them separately

1 $$= \sum_{j=0}^{N-1} (x_{j+1} - x_j)^2$$

$$= \sum_{j=0}^{N-1} \left[\frac{1}{\sqrt{N}} \sum_{k=-M}^{M} y_k \left(e^{i \frac{2\pi k}{N}(j+1)} - e^{i \frac{2\pi k}{N} j} \right) \right]^2$$

$$= \sum_{j=0}^{N-1} \left[\frac{1}{\sqrt{N}} \sum_{k=-M}^{M} y_k \, e^{i \frac{2\pi k}{N}(j + \frac{1}{2})} \left(e^{i \frac{\pi k}{N}} - e^{-i \frac{\pi k}{N}} \right) \right]^2$$

$$= \sum_{j=0}^{N-1} \cdot \frac{1}{N} \cdot \sum_{k=-M}^{M} \sum_{k'=-M}^{M} y_k y_{k'} \, e^{i \frac{2\pi}{N}(k+k')(j+\frac{1}{2})}$$

$$\times -4 \sin\left(\frac{\pi k}{N}\right) \sin\left(\frac{\pi k'}{N}\right)$$

$$= \cancel{\frac{1}{N}} \cdot \sum_{k=-M}^{M} \sum_{k'=-M}^{M} y_k y_{k'} \cdot \cancel{N} \delta_{k+k'} \cdot e^{i \frac{\pi}{N}(k+k')}$$

$$\times -4 \sin\left(\frac{\pi k}{N}\right) \sin\left(\frac{\pi k'}{N}\right)$$

$$= \sum_{k=-M}^{M} y_k y_{-k} \cdot 4 \sin\left(\frac{\pi k}{N}\right)^2$$

2 $$= \sum_{j=0}^{N-1} x_j^2 = \sum_{j=0}^{N-1} \left(\frac{1}{\sqrt{N}} \sum_{k=-M}^{M} e^{i \frac{2\pi k}{N} j} y_k \right)$$

$$= \sum_{j=0}^{N-1} \sum_{k=-M}^{M} \sum_{k'=-M}^{M} \cdot \frac{1}{N} \, e^{i \frac{2\pi j}{N}(k+k')} y_k y_{k'}$$

$$= \frac{1}{N} \sum_{k=-M}^{M} \sum_{k'=-M}^{M} \cdot N \delta_{k+k'} \, y_k y_{k'} = \sum_{k=-M}^{M} y_k y_{-k}$$

Together

$$S_E^{disc} = \beta \frac{Nm}{2\beta^2} \cdot (A) + \frac{\beta}{N} \cdot \frac{1}{2} m\omega^2 (B)$$

$$= \sum_{k=-M}^{M} y_k y_{-k} \left(\frac{Nm}{2\beta} \cdot 4 \sin\left(\frac{\pi k}{N}\right)^2 + \frac{\beta}{N} \cdot \frac{1}{2} m \omega^2 \right)$$

$$= \sum_{k=-M}^{M} y_k \cdot \frac{Nm}{2\beta} \left(4 \sin\left(\frac{\pi k}{N}\right)^2 + \frac{\beta^2 \omega^2}{N^2} \right) y_{-k}$$

$$= \sum_{k=-M}^{M} y_k \, A(k) \, y_k^*$$

where

$$A_k = \frac{Nm}{2\beta} \left(4 \sin\left(\frac{\pi k}{N}\right)^2 + \frac{\beta^2 \omega^2}{N^2} \right)$$

This is now diagonalized!

b) Partition function is

$$Z(\beta) = \left(\frac{m}{2\pi \varepsilon} \right)^{ND/2} \cdot \left(\prod_{i=0}^{N-1} \int d^D x_i \right) e^{-S_E}$$

Switch coordinates $x^i \to y^i$. Need Jacobian

$$|J| = |\det (u^i{}_j)| = \left| \det \left(\frac{\partial x^i}{\partial y^j} \right) \right|$$

To compute this, note that

$$\frac{\partial x^m}{\partial y^n} = \frac{1}{\sqrt{N}} \cdot \sum_{k=-M}^{M} \delta_n^k \cdot e^{i \frac{2\pi k}{N} m}$$

$$= \frac{1}{\sqrt{N}} e^{i \cdot \frac{2\pi}{N} \cdot m \cdot n}$$

Then

$$(u \cdot u^T)^m{}_n = u^m{}_k (u^T)^k{}_n = \frac{1}{N} \cdot \sum_{k=} e^{i \frac{2\pi}{N}(mk + kn)}$$

$$= \delta_{mn} = (I)_{mn}$$

so this is an orthogonal transformation. Then

$$1 = \det(I) = \det(UU^T) = \det(U)^2$$

$$\det(U) = \pm 1$$

so that $\underline{J = 1}$. Then

$2M+1 = N$

$$Z(\beta) = \left(\frac{m}{2\pi\varepsilon}\right)^{ND/2} \cdot \left(\prod_{k=-M}^{M} \int d^D \eta_k\right) \exp\left[-\sum \eta_k A(k) \eta_k^*\right]$$

$$= \left(\frac{m}{2\pi\varepsilon}\right)^{ND/2} \cdot \left(\frac{\pi^N}{\det A}\right)^{D/2}$$

$$= \left(\frac{m}{2\varepsilon}\right)^{ND/2} \cdot \left(A(0) \prod_{k=1}^{M} A(k) A(-k)\right)^{-D/2}$$

However $A(k) = A(-k)$,

$$= \left(\frac{m}{2\varepsilon}\right)^{ND/2} \cdot \left(\frac{1}{\sqrt{A(0)} \prod_{k=1}^{M} A(k)}\right)^D$$

we can now take the continuum limit $N \to \infty$ ($M \to \infty$

$$\to \lim_{N \to \infty} \left(\frac{m}{2\varepsilon}\right)^{ND/2} \cdot \left(\frac{1}{\sqrt{A(0)} \prod_{k=1}^{M} A(k)}\right)^D$$

i) Let's evaluate this for $N \to \infty$

$$A(k) = \frac{Nm}{2\beta}\left(4 \sin\left(\frac{\pi k}{N}\right)^2 + \frac{\beta^2 \omega^2}{N^2}\right).$$

$$\sim \frac{Nm}{2\beta}\left(4 \cdot \left(\frac{\pi k}{N}\right)^2 + \frac{\beta^2\omega^2}{N^2}\right) + O\left(\frac{1}{N^2}\right)$$

$$\sim \frac{Nm}{2\beta} \cdot \frac{4\pi^2 k^2}{N^2}\left(1 + \frac{\beta^2\omega^2}{N^2} \cdot \frac{N^2}{4\pi^2 k^2}\right)$$

$$\sim \frac{2m\pi^2 k^2}{\beta N}\left(1 + \frac{\beta^2\omega^2}{4\pi^2 k^2}\right)$$

Hence, using Euler's formula

$$\lim_{N\to\infty} \prod_{k=1}^{M} A(k) = \lim_{N\to\infty} \prod_{k=1}^{M} \frac{2m\pi^2 k^2}{\beta N} \left(1 + \frac{(\beta\omega/2)^2}{\pi^2 k^2} \right)$$

$$= \lim_{N\to\infty} \left(\prod_{k=1}^{M} \frac{2m\pi^2 k^2}{\beta N} \right) \cdot \frac{1}{\beta\omega/2} \cdot \sinh(\beta\omega/2)$$

Also

$$A(0) = \frac{Nm}{2\beta} \cdot \frac{\beta^2\omega^2}{N^2} = \frac{m\omega^2\beta}{2N}$$

Then

$$Z(\beta) = \lim_{N\to\infty} \left(\frac{m}{2\varepsilon} \right)^{ND/2} \left(\frac{1}{\sqrt{A(0)} \prod_{k=1}^{M} A(k)} \right)^{D}$$

$$= \lim_{N\to\infty} \left(\frac{mN}{2\beta} \right)^{ND/2} \left[\sqrt{\frac{m\omega^2 \cancel{\beta} 2}{\beta \cancel{2} N}} \cdot \frac{\sinh(\beta\omega/2)}{\cancel{\beta}\cancel{\omega}/\cancel{2}} \right.$$

$$\left. \times \prod_{k=1}^{M} \left(\frac{2m\pi^2 k^2}{\beta N} \right) \right]^{-D}$$

$$= \lim_{N\to\infty} \left(\frac{mN}{2\beta} \right)^{MD + \frac{D}{2}} \cdot \left(\frac{\beta N}{2m} \right)^{MD + \frac{D}{2}} \left(\prod_{k=1}^{M} (\pi k)^{-2} \right)^{D}$$

$$\cdot \frac{1}{\sinh(\beta\omega/2)^{D}}$$

$$= \lim_{N\to\infty} \left(\frac{N^2}{4} \right)^{MD + D/2} \cdot \left(\prod_{k=1}^{M} (\pi k)^{-2} \right)^{D} \frac{1}{\sinh(\beta\omega/2)^{D}}$$

$$= \lim_{N\to\infty} \left(\prod_{k=1}^{M} \left(\frac{2\pi k}{N} \right)^{-2} \right)^{D} \cdot \left(\frac{N}{2\sinh(\beta\omega/2)} \right)^{D}$$

Can use ζ-function regularisation to show that

$$\prod_{k=1}^{\infty} \left(\frac{2\pi k}{N} \right)^{-2} = \frac{1}{N}$$

To see this, recall that

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

Then, define

$$\tilde{\zeta}(s) = \left(\frac{N}{2\pi} \right)^{2s} \zeta(2s) = \sum_{n=1}^{\infty} \left(\frac{2\pi n}{N} \right)^{-2s}$$

Then, note that

$$\xi'(s) = \left(\frac{N}{2\pi}\right)^{2s} \cdot 2\log\left(\frac{N}{2\pi}\right) \zeta(2s)$$

$$+ \left(\frac{N}{2\pi}\right)^{2s} \cdot 2\zeta'(2s)$$

$$= 2\left(\frac{N}{2\pi}\right)^{2s}\left(\zeta'(2s) + \log\left(\frac{N}{2\pi}\right)\zeta(2s)\right)$$

and we can also write

$$= \sum_{n=1}^{\infty} \left(\frac{2\pi n}{N}\right)^{-2s} \cdot -2\log\left(\frac{2\pi n}{N}\right)$$

$$= -2\sum_{n=1}^{\infty}\left(\frac{2\pi n}{N}\right)^{-2s}\log\left(\frac{2\pi n}{N}\right)$$

Evaluating this on both sides at $s=0$,

$$2\left[\xi'(0) + \log\left(\frac{N}{2\pi}\right)\zeta(0)\right] = \sum_{n=1}^{\infty}\log\left[\left(\frac{2\pi n}{N}\right)^{-2}\right]$$

Taking exponentials

$$\prod_{n=1}^{\infty}\left(\frac{2\pi n}{N}\right)^{-2} = e^{2\zeta'(0)} \cdot \left(\frac{N}{2\pi}\right)^{2\zeta(0)}$$

$$= e^{2\cdot -\frac{1}{2}\log(2\pi)} \cdot \left(\frac{N}{2\pi}\right)^{2\cdot -\frac{1}{2}}$$

$$= \frac{1}{2\pi}\cdot\frac{2\pi}{N} = \frac{1}{N}$$

Hence, we find that

$$Z(\beta) = \left(\frac{1}{2\sinh(\beta\omega/2)}\right)^{D}$$

Note: In QM we can calculate and care about the constants. In QFT, we don't!

ii) Free energy

$$F = -\frac{1}{\beta} \log Z = \frac{D}{\beta} \log \left(2 \sinh \frac{\beta \omega}{2} \right)$$

At high temperatures, $\beta \to 0$, and

$$F \sim \frac{D}{\beta} \log \left(2 \cdot \frac{\beta \omega}{2} + \dots \right) \sim \frac{D}{\beta} \log (\beta \omega)$$

$$= F_{cl}$$

and we recover the classical result!

At low temperatures, $\beta \to \infty$,

$$F \sim \frac{D}{\beta} \log \left(2 \cdot \frac{e^{\omega \beta / 2} + \dots}{2} \right) = \frac{D}{\beta} \cdot \frac{\omega \beta}{2}$$

$$\sim D \cdot \frac{\omega}{2}$$

This is a quantum contribution - ground state energy for D-dimensional quantum harmonic oscillator!