AQFT: problem sheet 3

1. scalar field theory with N_s scalars ϕ_I, and action

$$S[\phi] = \int d^Dx \Big[\underbrace{\overbrace{-\tfrac{1}{2}(\partial\phi_I)^2 - \tfrac{1}{2}(m^2 - i\varepsilon)\phi_I^2}^{(-+\cdots+)}}_{S_0} \underbrace{- \frac{g_{IJK}}{3!}\phi_I\phi_J\phi_K}_{S_{int}} \Big]$$

$I, J \in \{1, \ldots, N_s\}$ with Einstein summation convention.

The partition/generating functional is defined as

$$Z[J] = \int D\phi \, \exp\left(iS[\phi] + i\int d^Dx \, J_I \phi_I \right)$$

$$= \exp\left(iS_{int}\left[\frac{1}{i}\frac{\delta}{\delta J_I}\right]\right) Z_0[J]$$

where, using previous results

$$Z_0[J] = \int D\phi \, \exp\left(iS_0[\phi] + i\int d^Dx \, J_I\phi_I \right)$$

$$= \int D\phi \, \exp\left(i\int d^Dx \, \tfrac{1}{2}\phi_I(\Box - m^2 + i\varepsilon)\phi_I + J_I\phi_I \right)$$

$$= Z_0[0] \exp\left[-\tfrac{1}{2}\int d^Dx \int d^Dy \, J_I(x) G_F(x,y) J_I(y) \right]$$

and we identify

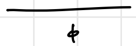
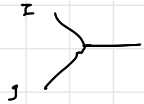
$$G_{IJ} = \frac{1}{Z_0[0]}\left(\frac{1}{i}\frac{\delta}{\delta J_I(x)}\right)\left(\frac{1}{i}\frac{\delta}{\delta J_J(y)}\right) Z_0[J]$$

$$= G_F(x,y) \cdot \delta_{IJ}$$

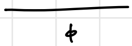
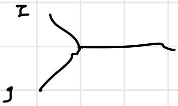
$$= \int d^Dk \, e^{ik\cdot(x-y)} \underbrace{\frac{-i\,\delta_{IJ}}{k^2 + m^2 - i\varepsilon}}_{\tilde{G}_{IJ}}$$

as the propagator.

All other Feynman rules just follow through directly, taking extra care to label the lines with field indices. In configuration space:

I  J $= G_{IJ}$  $= i\int d^Dx\, J_I(x)$  $k = -ig_{IJK}\int d^Dx$

or, after performing the integrals, in momentum space

I  J $= \tilde{G}_{IJ}$  $= i\tilde{J}_I$  $k = -ig_{IJK}$

To get the Wilson effective action

$$iW[J] = \log Z[J]$$

we take care to only consider connected diagrams.

2. The quantum / 1PI effective action

$$\Gamma[\Phi] = W[J] - \int d^Dx\, J(x)\,\Phi(x)$$

is the generating functional for 1PI diagrams $\Gamma^{(n)}$

$$\Gamma[\Phi] = \sum_{n=1}^{\infty}\frac{1}{n!}\left(\prod_{i=1}^{n}\int d^Dx_i\cdot\Phi(x_i)\right)\Gamma^{(n)}(x_1,\dots,x_n)$$

For $n=2$, we note that

$$\Gamma^{(2)}(x,y) = \frac{\delta\Gamma[\Phi]}{\delta\Phi(x)\,\delta\Phi(y)} = -\frac{\delta J(y)}{\delta\Phi(x)}$$

and

$$G_c^{(2)}(x,y) = i\left(\frac{1}{i}\right)^2\frac{\delta^2 W[J]}{\delta J(x)\,\delta J(y)} = -i\,\frac{\delta\Phi(y)}{\delta J(x)}$$

so that

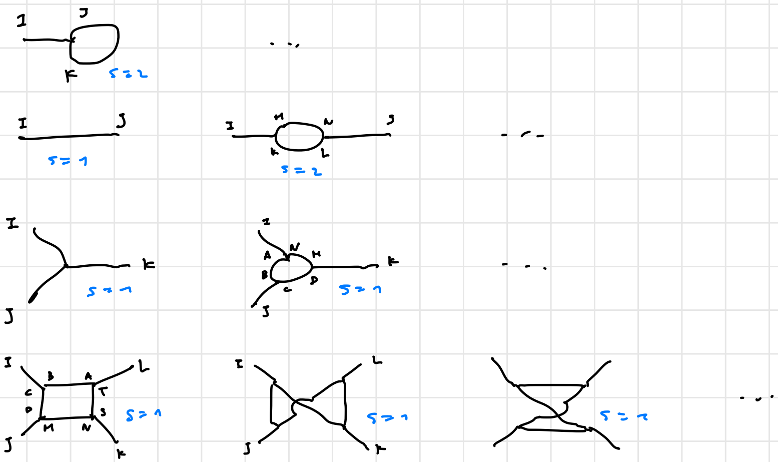
$$\int d^Dz\;\Gamma^{(2)}(x,z)\,G_c^{(2)}(z,y) = \int d^Dz\cdot -i\,\frac{\delta J(x)}{\delta\Phi(z)}\,\frac{\delta\Phi(z)}{\delta J(y)}$$

$$= -i\,\frac{\delta J(x)}{\delta J(y)} = -i\,\delta^{(D)}(x-y)$$

which just gives

$$\Gamma^{(2)} \sim -i\,[G^{(2)}]^{-1}$$

while for $n \geq 3$, $\Gamma^{(n)}$ are just 1PI vertices, up to $n=4$, we have



In particular, using amputated diagrams and then expanding

$i\Gamma[\bar{\phi}]$

$$= \frac{1}{1!}\left(\prod_{i=1}^{1}\int d^Dx_i\right)\bar{\phi}_I(x)\cdot\left[\frac{1}{2}(-i g_{IJK})\cdot G_{JK}(x_1,x_1)+\dots\right]$$

$$+\frac{1}{2!}\left(\prod_{i=1}^{2}\int d^Dx_i\right)\bar{\phi}_I(x_1)\,\bar{\phi}_J(x_2)\left[i\cdot\frac{\Box-m^2+i\varepsilon}{i}\cdot\delta_{IJ}\cdot\delta(x_1-x_2)\right.$$

$$\left.+\frac{1}{2}(-i g_{IKM})(-i g_{JNL})\,G_{MN}(x_1,x_2)\,G_{KL}(x_1,x_2)+\dots\right]$$

$$+\frac{1}{3!}\left(\prod_{i=1}^{3}\int d^Dx_i\right)\bar{\phi}_I(x_1)\,\bar{\phi}_J(x_2)\,\bar{\phi}_K(x_3)\left[-i g_{IJK}\right.$$

$$+(-i g_{IAN})(-i g_{JCB})(-i g_{KMD})\,G_{MN}(x_3,x_1)$$

$$\left.G_{AB}(x_1,x_2)\,G_{CD}(x_2,x_3)+\dots\right]$$

$$+ \frac{1}{4!} \left(\prod_{i=1}^{4} \int d^D x_i \right) \bar{\Phi}_I(x_1) \bar{\Phi}_J(x_2) \bar{\Phi}_K(x_3) \bar{\Phi}_L(x_4)$$

$$\Big[3 \, (-i g_{ICB})(-i g_{JMD})(-i g_{KSN})(-i g_{LAT})$$

$$G_{AB}(x_1, x_1) \, G_{CD}(x_1, x_2) \, G_{MN}(x_2, x_3) \, G_{ST}(x_3, x_4) + \ldots \Big]$$

$+ \ldots$

$$= i S[\bar{\Phi}] - \frac{i}{2} g^{IJJ} \bar{\Phi}^I_{x_1} G_{x_1 x_1} - \frac{1}{2 \cdot 2!} g^{IKL} g^{JLK} \bar{\Phi}^I_{x_1} G^2_{x_1 x_2} \bar{\Phi}^J_{x_2}$$

$$+ \frac{i}{3!} g^{IMN} g^{JLM} g^{KNL} \bar{\Phi}^I_{x_1} G_{x_1 x_2} \bar{\Phi}^J_{x_2} G_{x_2 x_3}$$

$$- \frac{1}{2^3} g^{IMN} g^{J4M} g^{KB4} g^{LNB} \bar{\Phi}^I_{x_1} G_{x_1 x_1} \bar{\Phi}^J_{x_2} G_{x_2 x_3} \bar{\Phi}^K_{x_3}$$

$$G_{x_3 4} \bar{\Phi}^L_{x_4} G_{x_4 x_1} + \ldots$$

3. Grassman algebra Λ_N with elements η_i, $i, j \in \{1, \ldots, N\}$ such that

$$\{\eta_i, \eta_j\} = 0$$

and using Berezin integration

$$\int d^N \eta \; \eta_{i_1} \ldots \eta_{i_N} = \int d\eta_N \ldots d\eta_1 \, \eta_{i_1} \ldots \eta_{i_N} = \varepsilon_{i_1 \ldots i_N}$$

consider the toy model for a interacting fermion partition function

$$Z = \int d^N \eta \, \exp\left[\frac{1}{2} \eta_i A_{ij} \eta_j + g B_{ijkl} \eta_i \eta_j \eta_k \eta_l \right]$$

where A, B are totally antisymmetric.

$$= \int d^N \eta \left[1 + \frac{1}{1!} \cdot \frac{1}{2} \eta_i A_{ij} \eta_j + \frac{1}{2!} \cdot \frac{1}{2^2} (\eta_i A_{ij} \eta_j)^2 + \ldots \right]$$

$$\times \left[1 + g B_{ijkl} \eta_i \eta_j \eta_k \eta_l + \frac{1}{2!} g^2 (B_{ijkl} \eta_i \eta_j \eta_k \eta_l)^2 + \ldots \right]$$

only the term with N powers of η survive!

For $\underline{N=4}$: B is top. form, so

$$B_{ijkl} = B\, \varepsilon_{ijkl}$$

and

$$Z = \left(\frac{1}{2!} \cdot \frac{1}{2^2} A_{ij} A_{kl} + g\, B_{ijkl} \right) \varepsilon_{ijkl}$$

$$= Pf(A) + 4!\, B$$

4. For $\underline{N=8}$:

$$Z = \varepsilon_{i_1 \ldots i_8} \left(\frac{1}{4!} \cdot \frac{1}{2^4} A_{i_1 i_2} A_{i_3 i_4} A_{i_5 i_6} A_{i_7 i_8} \right.$$

$$\left. + \frac{1}{2!} \cdot \frac{1}{2^2} A_{i_1 i_2} A_{i_3 i_4} \cdot g\, B_{i_5 \ldots i_8} + \frac{1}{2!} g^2 B_{i_1 \ldots i_4} B_{i_5 \ldots i_8} \right)$$

5. For $N \to \infty$ even, up to $O(g^2)$

$$Z = \int d^N \eta \; e^{\frac{1}{2} \eta_i A_{ij} \eta_j} \left[1 + g\, B_{ijkl} \eta_i \eta_j \eta_k \eta_l \right.$$

$$\left. + \frac{1}{2!} g^2 \left(B_{ijkl} \eta_i \eta_j \eta_k \eta_l \right)^2 \right]$$

$$= \varepsilon_{i_1 \ldots i_N} \left(\frac{1}{2^{N/2} \left(\frac{N}{2}\right)!} A_{i_1 i_2} \cdots A_{i_{N-1} i_N} \right.$$

$$+ \frac{1}{2^{(N-4)/2} \left(\frac{N-4}{2}\right)!} A_{i_1 i_2} \cdots A_{i_{N-5} i_{N-4}} \cdot g\, B_{i_{N-3} \ldots i_N}$$

$$+ \frac{1}{2^{(N-6)/2} \left(\frac{N-6}{2}\right)!} A_{i_1 i_2} \cdots A_{i_{N-9}} A_{i_{N-8}}$$

$$\left. \cdot \frac{g^2}{2!} B_{i_{N-7} \ldots i_{N-4}} B_{i_{N-3} \ldots i_N} \right)$$

6. Defining

$$\delta(\eta) = \eta_1 \cdots \eta_N$$

gives

$$\int d^N\eta \quad \delta(\eta) = \int d\eta_N \cdots d\eta_1 \quad \eta_1 \cdots \eta_N = 1$$

as required.

7. For a function $f(\eta)$,

$$\int d^N\eta \quad f(\eta) \quad \delta(\eta)$$

$$= \int d^N\eta \quad \delta(\eta) \cdot \left(\sum_{n=0}^{N} \frac{\partial^n f}{\partial\eta_{i_1} \cdots \partial\eta_{i_n}}\Big|_{\eta=0} \eta_{i_1} \cdots \eta_{i_n} \right)$$

Picks out the η^N term

$$= f(0)$$

8. Introduce Grassman odd sources J_i, such that

$$\{J_i, J_j\} = 0 \qquad \{J_i, \eta_j\} = 0$$

Define Fourier transform of f as

$$\tilde{f}(J) = \int d^N\eta \quad e^{-i\eta_i J_i} \quad f(\eta)$$

For consistency

$$f(\eta) = \int d^N\eta' \quad \delta(\eta' - \eta) \; f(\eta')$$

$$= \int d^N\eta' \quad (\eta' - \eta)_1 \cdots (\eta' - \eta)_N \quad f(\eta')$$

$$= \int d^N\eta' \quad \frac{1}{N!} (\eta' - \eta)_{i_1} \cdots (\eta' - \eta)_{i_N} \; \varepsilon_{i_1 \cdots i_N} \quad f(\eta')$$

$$= \int d^N\eta' \int d^N J \quad \frac{1}{N!} (\eta' - \eta)_{i_1} \cdots (\eta' - \eta)_{i_N} \; J_{i_1} \cdots J_{i_N} \, f(\eta')$$

$$= \int d^N\eta' \int d^N J \quad \frac{(-)^{\sum_{n=1}^{N} n}}{N!} \quad (\eta' - \eta)_{i_1} J_{i_1} \cdots (\eta' - \eta)_{i_N} J_{i_N} \, f(\eta')$$

$$= \int d^n \eta' \int d^n \xi \; \frac{(-1)^{n(n+1)/2}}{(-i)^n} \, e^{-i \xi_i (\eta' - \eta)_i} \, f(\eta')$$

$$= (-1)^{n(n+1)/2} \, i^n \cdot \int d^n \xi \; e^{i \xi_i \eta_i} \, \tilde{f}(\xi)$$

so

$$c = (-1)^{n(n+1)/2} \, i^n$$