AQFT: Problem Sheet 5

Note: Can derive "master formula" for loop integrals

$$\int d^d \ell \, \frac{(\ell^2)^a}{(\ell^2+\Delta)^b} = i \, \frac{\Gamma(b-a-d/2)\,\Gamma(a+d/2)}{(4\pi)^{d/2}\,\Gamma(b)\,\Gamma(d/2)} \, \Delta^{a-b+d/2}$$

and will need Feynman parametrisation

$$\frac{1}{A_1 \cdots A_n} = \int dF_n \left[\sum_{i=1}^{n} A_i x_i \right]^{-n}$$

where

$$dF_n = (n-1)! \int dx_1 \cdots \int dx_n \; \delta\Big(1 - \sum_{i=1}^{n} x_i\Big)$$

is normalised such that

$$\int dF_n = 1$$

1. Consider a single real scalar field with bare action

$$S = \int d^d x \left[-\frac{1}{2}(\partial\phi_b)^2 - \frac{1}{2} m_b^2 \phi_b^2 - \frac{g_b}{3!}\phi_b^3 - Y_b \phi_b \right]$$

with $d = 6-\epsilon$. This is equivalent to the renormalized action

$$= \int d^d x \Big[-\frac{1}{2} Z_\phi (\partial\phi_r)^2 - \frac{1}{2} Z_m m_r^2 \phi_r^2 - \frac{1}{3!} Z_g g_r \mu^{\epsilon/2} \phi_r^3$$

$$- Z_Y \mu^{4-\epsilon/2} Y_r \phi_r \Big]$$

Defining $Z_i = 1 + \delta Z_i$

$$= \int d^d x \Big[\overbrace{\frac{1}{2}\phi_r(\Box - m_r^2)\phi_r - \frac{1}{3!} g_r \mu^{\epsilon/2} \phi_r^3 - \mu^{4-\epsilon/2} Y_r \phi_r}^{S_r}$$

$$\left. \begin{aligned} &+ \frac{1}{2}\phi_r(\delta Z_\phi \Box - \delta Z_m m_r^2)\phi_r - \frac{1}{3!} g_r \delta Z_g \mu^{\epsilon/2} \phi_r^3 \\ &- \delta Z_Y \mu^{4-\epsilon/2} Y_r \phi_r \Big] \end{aligned} \right\} \begin{aligned} &S_{ct} \\ &\sim O(\hbar) \end{aligned}$$

By comparison

$$\phi_b = Z_\phi^{1/2}\,\phi_r$$

$$m_b^2\, Z_\phi = Z_m\, m_r^2$$

$$g_b \cdot Z_\phi^{3/2} = Z_g\, g_r\, \mu^{\epsilon/2}$$

Different choice!

$$\lambda_b Z_\phi^{2} = Z_\lambda\, \mu^{4-\epsilon}\, \lambda_r$$

From now on: drop renormalized subscripts!

Interested in 1PI effective action. By definition

$$\Gamma[\bar\phi] = W[J] - \int d^dx\; J(x)\,\bar\phi(x)$$

Then

$$e^{i\Gamma[\bar\phi]/\hbar} = e^{i\left(W[J] - \int d^dx\; J(x)\,\bar\phi(x)\right)/\hbar}$$

$$= \frac{\int D\phi\, \exp\left[\frac{i}{\hbar}\left(S[\phi] + \int d^dx\; J(x)\,\phi(x)\right)\right]}{\int D\phi\, \exp\left(\frac{i}{\hbar} S[\phi]\right)}\, e^{-\frac{i}{\hbar}\int d^dx\; J(x)\,\bar\phi(x)}$$

$$= Z[0]^{-1} \int D\phi\, \exp\left[\frac{i}{\hbar}\left(S[\phi] + \int d^dx\; J(x)\left(\phi(x) - \bar\phi(x)\right)\right)\right]$$

$$= Z[0]^{-1} \int D\phi\, \exp\left[\frac{i}{\hbar}\left(S[\phi] - \int d^dx\, \frac{\delta\Gamma[\bar\phi]}{\delta\bar\phi(x)}\left(\phi(x) - \bar\phi(x)\right)\right)\right]$$

Consider a loop expansion

$$\phi = \bar\phi + \sqrt{\hbar}\,\chi + \dots$$

$$\Gamma = \Gamma_0 + \hbar\,\Gamma_1 + \dots$$

Then

$$S[\phi] = S[\bar\phi] + \int d^dx \left.\frac{\delta S[\phi]}{\delta\phi(x)}\right|_{\phi=\bar\phi} \sqrt{\hbar}\,\chi(x)$$

$$+ \frac{1}{2}\int d^dx\, d^dy \left.\frac{\delta^2 S[\phi]}{\delta\phi(x)\,\delta\phi(y)}\right|_{\phi=\bar\phi} \sqrt{\hbar}\,\chi(x)\cdot\sqrt{\hbar}\,\chi(y)$$

$$+ \dots$$

$$= S[\bar\phi] + \sqrt{\hbar}\cdot\int d^dx \left.\frac{\delta S[\phi]}{\delta\phi(x)}\right|_{\phi=\bar\phi} \cdot\chi(x)$$

$$+ \frac{\hbar}{2}\int d^dx\, d^dy \left[Z_\phi \partial_y^2 - Z_m m^2 - Z_g g \mu^{\varepsilon/2}\bar\phi(y)\right]$$

$$\delta^{(d)}(x-y)\cdot\chi(x)\,\chi(y) + \dots$$

$$= S[\bar\phi] + \sqrt{\hbar}\int d^dx \left.\frac{\delta S[\phi]}{\delta\phi(x)}\right|_{\phi=\bar\phi} \chi(x)$$

$$+ \hbar\int d^dx \left[\frac{1}{2}\chi\left(Z_\phi \partial^2 - Z_m m^2\right)\chi - \frac{1}{2} Z_g g \mu^{\varepsilon/2}\bar\phi\,\chi^2\right]$$

$$+ \dots$$

Then

$$e^{i\Gamma[\bar\phi]/\hbar} = e^{\frac{i}{\hbar}\left(\Gamma_0 + \hbar\Gamma_1 + \dots\right)}$$

$$= Z[0]^{-1}\int \mathcal{D}\chi\, \exp\Bigg[\frac{i}{\hbar}\Bigg(S[\bar\phi] + \underline{\sqrt{\hbar}\cdot\int d^dx \left.\frac{\delta S[\phi]}{\delta\phi(x)}\right|_{\phi=\bar\phi} \chi(x)}$$

$$+ \frac{\hbar}{2}\int d^dx \left[\chi\left(Z_\phi \partial^2 - Z_m m^2\right)\chi - Z_g g \mu^{\varepsilon/2}\bar\phi\,\chi^2\right] + \dots$$

$$\underline{- \int d^dx \left[\frac{\delta\Gamma_0[\bar\phi]}{\delta\bar\phi(x)}\right.} \left. + \hbar\,\frac{\delta\Gamma_1[\bar\phi]}{\delta\bar\phi(x)} + \dots\right]\cdot\sqrt{\hbar}\,\chi(x)\Bigg)\Bigg]$$

At leading order,

$$\Gamma_0[\bar\phi] = S_r[\bar\phi]$$

Physically, this is because 1PI vertices at $\underline{\text{tree-level}}$ are just the naive vertices. Then, at next-order, i.e. $\underline{\text{1-loop}}$

$$e^{i\Gamma_1[\bar\phi]} = Z[0]^{-1}\int \mathcal{D}\chi \exp\Big[\frac{i}{\hbar}\Big(S_{ct}[\bar\phi]$$

$$+ \sqrt{\hbar}\int d^dx \cancel{\frac{\delta S[\phi]}{\delta\phi(x)}}\Big|_{\phi=\bar\phi}\chi(x) - \sqrt{\hbar}\int d^dx \cancel{\frac{\delta\Gamma_0[\bar\phi]}{\delta\bar\phi(x)}}\chi(x)$$

$$+ \hbar\int d^dx \Big[\frac{1}{2}\chi(Z_\phi\Box - Z_m m^2)\chi - \frac{1}{2}Z_g g\mu^{\epsilon/2}\bar\phi\chi^2\Big]\Big)\Big]$$

Since $Z_\phi = 1 + O(\hbar)$, then at this order

$$= Z[0]^{-1} e^{iS_{ct}[\bar\phi]}$$

$$\int \mathcal{D}\chi \exp\Big(\frac{i}{2}\int d^dx\,[\chi(\Box - m^2)\chi - g\mu^{\epsilon/2}\bar\phi\chi^2]\Big)$$

Now, when the sources or currents off $J = 0$ and $\bar\phi = 0$, so

$$= e^{iS_{ct}[\bar\phi]}\frac{\int \mathcal{D}\chi \exp\Big(\frac{i}{2}\int d^dx\,[\chi(\Box - m^2)\chi - g\mu^{\epsilon/2}\bar\phi\chi^2]\Big)}{\int \mathcal{D}\chi \exp\Big(\frac{i}{2}\int d^dx\,[\chi(\Box - m^2)\chi]\Big)}$$

and using Gaussian integration

$$= e^{iS_{ct}[\bar\phi]}\frac{\sqrt{\det(\Box - m^2)}}{\sqrt{\det(\Box - m^2 - g\mu^{\epsilon/2}\bar\phi)}}$$

then

$$\Gamma_1 - S_{ct} = -i\ln\left\{\sqrt{\frac{\det(\Box - m^2)}{\det(\Box - m^2 - g\mu^{\epsilon/2}\bar\phi)}}\right\}$$

$$= \frac{i}{2}\ln\left\{\frac{\det(\Box - m^2 - g\mu^{\epsilon/2}\bar\phi)}{\det(\Box - m^2)}\right\}$$

$$= \frac{i}{2}\ln\left\{\det\left(\frac{\Box - m^2 - g\mu^{\epsilon/2}\bar\phi}{\Box - m^2}\right)\right\}$$

$$= \frac{i}{2}\mathrm{Tr}\left\{\ln\left(1 - \frac{g\mu^{\epsilon/2}\bar\phi}{\Box - m^2}\right)\right\}$$

use basis of Hilbert space to compute trace

$$= \frac{i}{2}\mathrm{Tr}\left\{\ln\left(1 + \frac{g\mu^{\epsilon/2}\bar\phi}{\partial_\mu\partial^\mu + m^2}\right)\right\}$$

Together, up to 1-loop

$$\Gamma[\bar\phi] = S_r[\bar\phi] + S_{ct}[\bar\phi] + \frac{i\hbar}{2} \mathrm{Tr}\left\{ \ln\left(1 + \frac{g\mu^{\varepsilon/2}\bar\phi(\hat x)}{\hat\partial_\mu \hat\partial^\mu + m^2}\right)\right\}$$

$$= S[\bar\phi] + \frac{i\hbar}{2} \mathrm{Tr}\left\{ \ln\left(1 + \frac{g\mu^{\varepsilon/2}\bar\phi(\hat x)}{\hat\partial_\mu \hat\partial^\mu + m^2}\right)\right\}$$

2. Interested in tadpole, so look at terms linear in $\bar\phi$

$$\Gamma[\bar\phi] \supset - \int d^d x \; Z_\nu \, \nu \, \mu^{4-\varepsilon/2} \bar\phi + \frac{i}{2} \mathrm{Tr}\left(\frac{g\mu^{\varepsilon/2}\bar\phi(\hat x)}{\hat\partial^\mu \hat\partial_\mu + m^2}\right)$$

where

$$\mathrm{Tr}\left(\frac{g\mu^{\varepsilon/2}\bar\phi(\hat x)}{\hat\partial^\mu \hat\partial_\mu + m^2}\right) = \int d^d x \, d^d k \; \langle x|k\rangle \langle k| \frac{g\mu^{\varepsilon/2}\bar\phi(\hat x)}{\hat\partial_\mu \hat\partial^\mu + m^2} |x\rangle$$

$$= g\mu^{\varepsilon/2} \int d^d x \; \bar\phi(x) \int d^d k \; \frac{1}{k^2+m^2} \langle \cancel{x|k}\rangle \langle \cancel{k|x}\rangle$$

$$= g\mu^{\varepsilon/2} \int d^d x \; \bar\phi(x) \int d^d k \; \frac{1}{k^2+m^2}$$

using the master integral with $a = 0$, $b = 1$, $\Delta = m^2$

$$\int d^d k \; \frac{1}{k^2+m^2} = i \frac{\Gamma(1-d/2)\,\cancel{\Gamma(d/2)}}{(4\pi)^{d/2}\,\cancel{\Gamma(1)}\,\cancel{\Gamma(d/2)}} (m^2)^{-1+d/2}$$

$$= i \cdot \frac{(m^2)^{2-\varepsilon/2}}{(4\pi)^{3-\varepsilon/2}} \Gamma\left(-2+\frac{\varepsilon}{2}\right)$$

$$= i \cdot \frac{m^4}{(4\pi)^3} \cdot \mu^{-\varepsilon} \left(\frac{4\pi\mu^2}{m^2}\right)^{\varepsilon/2} \left(\frac{1}{\varepsilon} + \frac{3}{4} - \frac{\gamma}{2} + O(\varepsilon)\right)$$

$$= i \cdot \frac{m^4}{(4\pi)^3} \cdot \mu^{-\varepsilon} \left[1 + \frac{\varepsilon}{2}\ln\left(\frac{4\pi\mu^2}{m^2}\right) + O(\varepsilon)\right]$$

$$\cdot \left(\frac{1}{\varepsilon} + \frac{3}{4} - \frac{\gamma}{2} + O(\varepsilon)\right)$$

$$= \frac{i m^4}{(4\pi)^3} \mu^{-\varepsilon} \left[\frac{1}{\varepsilon} + \frac{3}{4} - \frac{\gamma}{2} + \frac{1}{2}\ln\left(\frac{4\pi\mu^2}{m^2}\right) + O(\varepsilon)\right]$$

Together,

$$\Gamma[\bar\phi] \supset \int d^d x \; \bar\phi(x) \, T^{(1)}(x)$$

where

$$\Gamma^{(1)}(x) = -Z_\nu \nu \mu^{4-\varepsilon/2} - \frac{1}{2} g \mu^{-\varepsilon/2} \frac{m^4}{(4\pi)^3}$$

$$\cdot \left[\frac{1}{\varepsilon} + \frac{3}{4} - \frac{\gamma}{2} + \frac{1}{2} \ln \left(\frac{4\pi \mu^2}{m^2} \right) + O(\varepsilon^2) \right]$$

$$= \mu^{-\varepsilon/2} \left[- Z_\nu \nu \mu^4 - \frac{1}{2} g \frac{m^4}{(4\pi)^3} \cdot \left(\frac{1}{\varepsilon} + \frac{3}{4} - \frac{\gamma}{2} \right.\right.$$

$$\left.\left. + \frac{1}{2} \ln \left(\frac{4\pi\mu^2}{m^2} \right) \right) + O(\varepsilon) \right]$$

For this to vanish, we need

$$Z_\nu \nu \mu^4 = (1 + \delta Z_\nu) \nu \mu^4$$

$$= - \frac{1}{2} g \mu^{-\varepsilon/2} \frac{m^4}{(4\pi)^3} \left[\overbrace{\underbrace{\frac{1}{\varepsilon}}_{MS} + \frac{3}{4} - \frac{\gamma}{2} + \frac{1}{2} \ln \left(\frac{4\pi\mu^2}{m^2} \right)}^{\overline{MS}} \right.$$

$$\left. + O(\varepsilon) \right]$$

In MS scheme, we need

$$\delta Z_\nu \nu = - \frac{1}{2} g \mu^{-\varepsilon/2} \frac{(m/\mu)^4}{(4\pi)^3} \cdot \frac{1}{\varepsilon}$$

$$\nu = - \frac{1}{2} g \mu^{-\varepsilon/2} \frac{(m/\mu)^4}{(4\pi)^3} \left[\frac{3}{4} - \frac{\gamma}{2} + \frac{1}{2} \ln \left(\frac{4\pi\mu^2}{m^2} \right) \right]$$

so that

$$\delta Z_\nu = \frac{1/\varepsilon}{\frac{3}{4} - \frac{\gamma}{2} + \frac{1}{2} \ln (4\pi\mu^2/m^2)}$$

3. Now expand Γ up to cubic order in $\bar{\phi}$ to determine Z_i:

$$\Gamma[\bar{\phi}] = S[\bar{\phi}] + \frac{i\hbar}{2} \mathrm{Tr} \left\{ \ln \left(1 + \frac{g \mu^{\varepsilon/2} \bar{\phi}(\bar{x})}{\bar{\partial}_\mu \bar{\partial}^\mu + m^2} \right) \right\}$$

$$= \int d^d x \left[\frac{1}{2} \bar{\phi} (Z_\phi \partial \cdot Z_m m^2) \bar{\phi} - \frac{1}{3!} Z_g g \mu^{\varepsilon/2} \bar{\phi}^3 \right.$$

$$\left. - Z_\nu \cancel{\nu \mu^{4-\varepsilon/2}} \bar{\phi} \right] + \frac{i}{2} \mathrm{Tr} \left[\cancel{\frac{g \mu^{\varepsilon/2} \bar{\phi}(\bar{x})}{\bar{\partial}^\mu \bar{\partial}_\mu + m^2}} \right.$$

$$\left. - \frac{1}{2} \left(\frac{g \mu^{\varepsilon/2} \bar{\phi}(\bar{x})}{\bar{\partial}^\mu \bar{\partial}_\mu + m^2} \right)^2 + \frac{1}{3} \left(\frac{g \mu^{\varepsilon/2} \bar{\phi}(\bar{x})}{\bar{\partial}^\mu \bar{\partial}_\mu + m^2} \right)^3 + \dots \right\}$$

Already cancelled the tadpole, so

$$= \int d^dx \left[\frac{1}{2} \bar\phi \left(Z_\phi \Box - Z_m m^2 \right) \bar\phi - \frac{1}{3!} Z_g g \mu^{\epsilon/2} \bar\phi^3 \right\}$$

$$- \frac{i}{4} \mu^\epsilon g^2 \, \mathrm{Tr} \left(\frac{\bar\phi(\hat x)}{\hat p^\mu \hat p_\mu + m^2} \right)^2 + \frac{i}{6} \mu^{3\epsilon/2} g^3 \, \mathrm{Tr} \left(\frac{\bar\phi(\hat x)}{\hat p^\mu \hat p_\mu + m^2} \right)^3 + \dots$$

Start with $\underline{\text{quadratic terms}}$. First note that

$$\Gamma[\bar\phi] \supset \int d^dx \, \frac{1}{2} \bar\phi(x) \left(Z_\phi \Box - Z_m m^2 \right) \bar\phi(x)$$

$$= \frac{1}{2} \int \underline{d^dx} \left(\int d^dp_1 \, \tilde\phi(p_1) \, \underline{e^{ip_1 \cdot x}} \right) \left(Z_\phi \Box - Z_m m^2 \right)$$

$$\left(\int d^dp_2 \, \tilde\phi(p_2) \, \underline{e^{ip_2 \cdot x}} \right)$$

$$= \frac{1}{2} \int d^dp_1 \, d^dp_2 \left(Z_\phi \cdot -p^2 - Z_m m^2 \right) \tilde\phi(p_1) \tilde\phi(p_2)$$

$$\delta^{(d)}(p_1 + p_2)$$

$$= \frac{1}{2} \left(\prod_{i=1}^{2} \int d^dp_i \, \tilde\phi(p_i) \right) \underbrace{\left(-Z_\phi p_2^2 - Z_m m^2 \right)}_{\subset \Gamma^{(2)}} \delta^{(d)}(p_1 + p_2)$$

Further,

$$i\Gamma[\bar\phi] \supset -\frac{i}{4} \mu^\epsilon g^2 \, \mathrm{Tr} \left(\frac{\bar\phi(\hat x)}{\hat p^\mu \hat p_\mu + m^2} \right)^2$$

naively ordering ambiguity, but trace is cyclic! ✓

$$= -\frac{i}{4} \mu^\epsilon g^2 \left(\prod_{i=1}^{2} \int d^dx_i \int d^dp_i \right) \langle x_1 | \bar\phi(\hat x) | p_1 \rangle$$

$$\langle p_1 | \frac{1}{\hat p^\mu \hat p_\mu + m^2} | x_2 \rangle \langle x_2 | \bar\phi(\hat x) | p_2 \rangle \langle p_2 | \frac{1}{\hat p_\mu \hat p^\mu + m^2} | x_1 \rangle$$

$$= -\frac{i}{4} \mu^\epsilon g^2 \left(\prod_{i=1}^{2} \underline{\int d^dx_i} \int d^dp_i \, \frac{\bar\phi(x_i)}{p_i^2 + m^2} \right)$$

$$e^{\underline{ip_1 \cdot x_1} \, \underline{- ip_1 \cdot x_2} \, \underline{+ ip_2 \cdot x_2} \, \underline{- ip_2 \cdot x_1}}$$

$$= -\frac{i}{4} \mu^\epsilon g^2 \left(\prod_{i=1}^{2} \int d^dp_i \, \frac{1}{p_i^2 + m^2} \right) \tilde\phi(p_1 - p_2) \, \tilde\phi(p_2 - p_1)$$

Define $p = p_1 - p_2$ and relabel $p_2 \to k$

$$= -\frac{i}{2}\mu^{\varepsilon} g^2 \int d^dk \int d^dp \frac{1}{(p+k)^2+m^2}\frac{1}{k^2+m^2}\bar{\Phi}(p)\bar{\Phi}(-p)$$

$$= -\frac{i}{2}\mu^{\varepsilon} g^2 \left(\prod_{i=1}^{2}\int d^dp_i\, \bar{\Phi}(p_i)\right)\delta^{(d)}(p_1+p_2)$$

$$\cdot \int d^dk \frac{1}{k^2+m^2}\frac{1}{(p+k)^2+m^2} \quad \swarrow \in \Gamma^{(2)}$$

Now,

$$\int d^dk \frac{1}{k^2+m^2}\frac{1}{(k+p)^2+m^2}$$

$$= \int d^dk \int dF_2 \left[x_1(k^2+m^2) + x_2\left((k+p)^2+m^2\right)\right]^{-2}$$

$$= \int d^dk \int dF_2 \left[(1-\underline{x_2})(k^2+m^2) + x_2\left((\underline{k}+p)^2\underline{+m^2}\right)\right]^{-2}$$

$$= \int d^dk \int dF_2 \left[k^2+m^2 + x_2(2k\cdot p + p^2)\right]^{-2}$$

$$= \int dF_2 \int d^dk \left[(k+x_2p)^2 + x_2(1-x_2)p^2 + m^2\right]^{-2}$$

using the master integral with $\ell = k + x_2 p$ and $a=0$, $b=2$, and $\Delta_2 = x_2(1-x_2)p^2+m^2$ gives

$$= \int dF_2 \; i\cdot\frac{\Gamma(2-d/2)}{\Gamma(2)(4\pi)^{d/2}}\Delta_2^{-2+d/2}$$

$$= i\int dF_2\; \Gamma\left(-1+\frac{\varepsilon}{2}\right)\cdot\frac{\Delta_2^{1-\varepsilon/2}}{(4\pi)^{3-\varepsilon/2}}$$

$$= i\int dF_2 \frac{\Delta_2}{(4\pi)^3}\cdot\left(\frac{4\pi\mu^2}{\Delta_2}\right)^{\varepsilon/2}\cdot\mu^{-\varepsilon}\;\Gamma(-1+\varepsilon/2)$$

$$= i\mu^{-\varepsilon}\int dF_2 \frac{\Delta_2}{(4\pi)^3}\cdot\left(1+\frac{\varepsilon}{2}\ln\left(\frac{4\pi\mu^2}{\Delta_2}\right)+O(\varepsilon)\right)$$

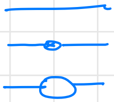
$$\left(-\frac{2}{\varepsilon} - 1 + \gamma + O(\varepsilon)\right)$$

$$= i\mu^{-\varepsilon}\int dF_2 \frac{\Delta_2}{(4\pi)^3}\left(-\frac{2}{\varepsilon} - 1 + \gamma - \ln\left(\frac{4\pi\mu^2}{\Delta_2}\right) + O(\varepsilon)\right)$$

Together, on quadratic order,

$$\Gamma[\bar\phi] \supset \frac{1}{2}\left(\prod_{i=1}^{2}\int d^d p_i\ \bar\phi(p_i)\right)\delta^{(d)}(p_1+p_2)\,\bar\Gamma^{(2)}(p_1,p_2)$$

$$= \frac{1}{2}\int d^d p\ \bar\phi(p)\,\bar\phi(-p)\ \bar\Gamma^{(2)}(p,-p)$$



where

$$\bar\Gamma^{(2)}(p,-p) = -(Z_\phi p^2 + Z_m m^2)$$

$$+\frac{1}{2}g^2\int dF_2\ \frac{\Delta_2}{(4\pi)^3}\left(\overbrace{\underbrace{\frac{2}{\epsilon}}_{MS} - 1 + \gamma - \ln\left(\frac{4\pi\mu^2}{\Delta_2}\right)}^{\overline{MS}} + O(\epsilon)\right)$$

In MS scheme, we want

$$\delta Z_\phi p^2 + \delta Z_m m^2$$

$$= \cancel{\frac{1}{2}}\,g^2\int dF_2\ \frac{\Delta_2}{(4\pi)^3}\ -\frac{\cancel{2}}{\epsilon}$$

$$= -\frac{1}{\epsilon}\,\frac{g^2}{(4\pi)^3}\cdot(2-1)!\cdot\int_0^1 dx_2\ \left(x_1(1-x_2)p^2 + m^2\right)$$

$$= -\frac{1}{\epsilon}\,\frac{g^2}{(4\pi)^3}\cdot\left(m^2 + \left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^1 p^2\right)$$

$$= -\frac{1}{\epsilon}\,\frac{g^2}{(4\pi)^3}\left(m^2 + \frac{1}{6}p^2\right)$$

so

$$\delta Z_\phi = -\frac{1}{\epsilon}\cdot\frac{g^2}{6(4\pi)^3} \qquad \delta Z_m = -\frac{1}{\epsilon}\,\frac{g^2}{(4\pi)^3}$$

Next, consider $\Gamma[\bar\phi]$ at cubic order in $\bar\phi$. First note that

$$\Gamma[\bar\phi] \supset -\frac{1}{3!}Z_g\,g\,\mu^{\epsilon/2}\int d^d x\ \bar\phi(x)^3$$

$$= -\frac{1}{3!}Z_g\,g\,\mu^{\epsilon/2}\left(\prod_{i=1}^{3}\int d^d p_i\ \bar\phi(p_i)\right)\delta^{(d)}\left(\sum_{i=1}^{3}p_i\right)$$

Further,

$$\Gamma[\bar\phi] \supset \frac{i}{6}\,\mu^{3\epsilon/2}\,g^3\ \mathrm{Tr}\left(\frac{\bar\phi(\hat x)}{\hat p^\mu\hat p_\mu + m^2}\right)^3$$

$$= \frac{i}{6} \mu^{3\varepsilon/2} g^3 \left(\prod_{i=1}^{3} \int d^d x_i \, d^d p_i \right) \langle x_1 | \bar{\phi}(\bar{x}) | p_1 \rangle$$

$$\langle p_1 | \frac{1}{\bar{p}^\mu \bar{p}_\mu + m^2} | x_1 \rangle \langle x_2 | \bar{\phi}(\bar{x}) | p_2 \rangle \langle p_2 | \frac{1}{\bar{p}_\mu \hat{p}^\mu + m^2} | x_3 \rangle$$

$$\langle x_3 | \bar{\phi}(\bar{x}) | p_3 \rangle \langle p_3 | \frac{1}{\bar{p}_\mu \hat{p}^\mu + m^2} | x_1 \rangle$$

$$= \frac{i}{6} \mu^{3\varepsilon/2} g^3 \left(\prod_{i=1}^{3} \int \underline{\underline{\underline{d^d x_i}}} \, d^d p_i \frac{\bar{\phi}(x_i)}{p_i^2 + m^2} \right)$$

$$e^{\underline{ip_1 \cdot x_1} \underline{- ip_1 \cdot x_2} + \underline{ip_2 \cdot x_2} \underline{- ip_2 \cdot x_3} + \underline{ip_3 \cdot x_3} - \underline{ip_3 \cdot x_1}}$$

$$= \frac{i}{6} \mu^{3\varepsilon/2} g^3 \left(\prod_{i=1}^{3} \int d^d p_i \frac{1}{p_i^2 + m^2} \right) \bar{\phi}(p_1 - p_3) \bar{\phi}(p_2 - p_1)$$

$$\bar{\phi}(p_3 - p_2)$$

Relabel $q_1 = p_1 - p_3$, $q_3 = p_3 - p_1$, $q_2 = p_1$

$$= \frac{i}{3!} \mu^{3\varepsilon/2} g^3 \left(\prod_{i=1}^{3} \int d^d q_i \right) \bar{\phi}(q_1) \bar{\phi}(-q_1 - q_3) \bar{\phi}(q_3)$$

$$\frac{1}{(q_1 + q_2 + q_3)^2 + m^2} \frac{1}{q_2^2 + m^2} \frac{1}{(q_2 + q_3)^2 + m^2}$$

and now relabel $k = q_2$, $p_1 = q_3$, $p_2 = q_1$, $p_3 = q_2$

$$= \frac{i}{3!} \mu^{3\varepsilon/2} g^3 \left(\prod_{i=1}^{3} \int d^d p_i \, \bar{\phi}(p_i) \right) \delta^{(d)}\left(\sum_{i=1}^{3} p_i \right)$$

$$\cdot \int d^d k \frac{1}{k^2 + m^2} \frac{1}{(k + p_1)^2 + m^2} \frac{1}{(k - p_3)^2}$$

Now,

$$\int d^d k \frac{1}{k^2 + m^2} \frac{1}{(k + p_1)^2 + m^2} \frac{1}{(k - p_3)^2 + m^2}$$

$$= \int d^d k \int dF_3 \cdot \Big[x_1 (k^2 + m^2) + x_2 ((k + p_1)^2 + m^2)$$

$$+ x_3 ((k - p_3)^2 + m^2) \Big]^{-3}$$

$$= \int d^d k \int dF_3 \Big[(1 - \cancel{x_2} - \cancel{x_3})(k^2 + m^2) + x_2 (\cancel{k^2} + 2k \cdot p_1 + p_1^2 \quad \overset{2}{m^2})$$

$$+ x_3 \left(\cancel{k^2} - 2k\cdot p_3 + p_3^2 \cancel{+ m^2} \right) \Big]^{-3}$$

$$= \int d^d k \int dF_3 \Big[k^2 + 2k\cdot(x_2 p_1 - x_3 p_3) + m^2$$

$$+ x_1 p_1^2 + x_3 p_3^2 \Big]^{-3}$$

$$= \int dF_3 \int d^d k \Big[(k + x_2 p_1 - x_3 p_3)^2 + 2 x_2 x_3 p_1\cdot p_3$$

$$+ x_2 (1 - x_2) p_1^2 + x_3 (1 - x_3) p_3^2 + m^2 \Big]^{-3}$$

Using the master integral with $\ell = k + x_2 p_1 - x_3 p_3$ and $a = 0$, $b = 3$, $\Delta_3 = m^2 + 2 x_2 x_3 p_1 \cdot p_3 + x_2 (1 - x_2) p_1^2 + x_3 (1 - x_3) p_1^2$

$$= \int dF_3 \cdot i \frac{\Gamma(3 - d/2)}{(4\pi)^{d/2} \Gamma(3)} \cdot \Delta_3^{-3 + d/2}$$

$$= \frac{i}{2} \int dF_3 \frac{\Gamma(\varepsilon/2)}{(4\pi)^{3-\varepsilon/2}} \Delta_3^{-\varepsilon/2}$$

$$= \frac{i}{2(4\pi)^3} \cdot \int dF_3 \left(\frac{4\pi\mu^2}{\Delta_3} \right)^{\varepsilon/2} \mu^{-\varepsilon} \Gamma(\varepsilon/2)$$

$$= \frac{i\mu^{-\varepsilon}}{2(4\pi)^3} \cdot \int dF_3 \left(1 + \frac{\varepsilon}{2} \ln \frac{4\pi\mu^2}{\Delta_3} + O(\varepsilon) \right)$$

$$\cdot \left(\frac{2}{\varepsilon} - \gamma + O(\varepsilon) \right)$$

$$= \frac{i\mu^{-\varepsilon}}{2(4\pi)^3} \int dF_3 \left(\frac{2}{\varepsilon} - \gamma + \ln \frac{4\pi\mu^2}{\Delta_3} + O(\varepsilon) \right)$$

Together, at cubic order

$$i\Gamma[\phi] = \frac{i}{3!} \left(\prod_{i=1}^{3} \int d^d p_i \, \phi(p_i) \right) \delta^{(d)}\left(\sum_{i=1}^{3} p_i \right) \tilde{\Gamma}^{(3)}(p_1, p_2, p_3)$$

where

$$\tilde{\Gamma}^{(3)}(p_1, p_2, p_3) = - Z_g g \mu^{\varepsilon/2}$$

$$- \frac{1}{2} \mu^{\varepsilon/2} \left(\frac{g}{4\pi} \right)^3 \int dF_3 \Big(\underbrace{\overbrace{\frac{2}{\varepsilon}}^{MS} - \gamma + \ln \frac{4\pi\mu^2}{\Delta_3}}_{\overline{MS}} + O(\varepsilon) \Big)$$

in MS scheme, we need

$$\delta Z_g\, g = \frac{1}{2}\left(\frac{g}{4\pi}\right)^3 \int dF_3 \;\frac{2}{\varepsilon} = -\frac{g^3}{(4\pi)^3}\cdot\frac{1}{\varepsilon}$$

4. After cancelling all the divergences, we are left with

$$\Gamma[\phi] = \sum_{n=1} \frac{1}{n!}\left(\prod_{i=1}^{n}\int d^d p_i \;\tilde{\phi}(p_i)\right)\delta^{(d)}\left(\sum_{i=1}^{n} p_i\right)$$

$$\times \;\tilde{\Gamma}^{(n)}(p_1,\dots,p_n)$$

where in MS scheme, we are left with the following finite parts

$$\tilde{\Gamma}^{(1)}(p) = 0$$

$$\tilde{\Gamma}^{(2)}(p,-p) = -p^2 - m^2 + \frac{1}{2}\frac{g^2}{(4\pi)^3}\int dF_2$$

$$\cdot\Delta_2\left(-1+\gamma-\ln\frac{4\pi\mu^2}{\Delta_2}\right)$$

$$\tilde{\Gamma}^{(3)}(p_1,p_2,p_3) = -\mu^{\varepsilon/2}\left[g + \frac{1}{2}\left(\frac{g}{4\pi}\right)^3\int dF_3\right.$$

$$\left.\left(-\gamma+\ln\frac{4\pi\mu^2}{\Delta_3}\right)\right]$$

etc.

5. The effective action with bare fields is independent of the renormalisation scale, μ

$$\frac{d}{d\ln\mu}\Gamma_b^{(n)}(x_1,\dots,x_n) = 0$$

Here

$$\Gamma_b^{(n)}(x_1,\dots,x_n) = \frac{\delta^n\Gamma[\phi]}{\delta\phi_b(x_1)\dots\delta\phi_b(x_n)}$$

$$= \left(\prod_{i=1}^{n}\int d^d y_i\;\frac{\delta\phi(y_i)}{\delta\phi_b(x_i)}\;\frac{\delta}{\delta\phi(y_i)}\right)\Gamma[\phi]$$

$$= \left(\prod_{i=1}^{n} \int d^4y_i \; Z_\phi^{-1/2} \cdot \delta(x_i - y_i) \cdot \right) \Gamma^{(n)}(y_1, \dots, y_n)$$

$$= Z_\phi^{-n/2} \, \Gamma^{(n)}(x_1, \dots, x_n)$$

so that

$$0 = \frac{d}{d\ln\mu}\left(Z_\phi^{-n/2} \, \Gamma^{(n)}(x_1, \dots, x_n) \right)$$

$$= -\frac{n}{2} Z_\phi^{-\frac{n}{2}-1} \frac{dZ_\phi}{d\ln\mu} \Gamma^{(n)} + \frac{d}{d\ln\mu} \Gamma^{(n)}$$

$$= Z_\phi^{-n/2} \left(-n \cdot \frac{1}{2} \frac{d\ln Z_\phi}{d\ln\mu} + \frac{\partial}{\partial\ln\mu} + \frac{dm^2}{d\ln\mu}\frac{\partial}{\partial m^2} + \frac{dg}{d\ln\mu}\frac{\partial}{\partial g} \right) \Gamma^{(n)}$$

$$= Z_\phi^{-n/2} \left(\frac{\partial}{\partial\ln\mu} + \beta_m \frac{\partial}{\partial m^2} + \beta_g \frac{\partial}{\partial g} - n \cdot \gamma_\phi \right) \Gamma^{(n)}$$

where

$$\gamma_\phi = \frac{1}{2} \frac{d\ln Z_\phi}{d\ln\mu}$$

✓ Different from Andrew!

6. Want to evaluate β-functions from the RG equations. Solve everything at $O(\hbar)$, so restore $\hbar$. At quadratic order

$$\frac{\partial\Gamma^{(2)}}{\partial\ln\mu} = \frac{\hbar}{2} \cdot \frac{g^2}{(4\pi)^3} \cdot \int dF_2 \; \Delta_2 \cdot -2$$

$$= -\hbar \frac{g^2}{(4\pi)^3} \cdot \int_0^1 dx \left(m^2 + x(1-x) p^2 \right)$$

$$= -\hbar \frac{g^2}{(4\pi)^3} \left(m^2 + \frac{1}{6} p^2 \right)$$

$$\frac{\partial\Gamma^{(2)}}{\partial m^2} = -1 + O(\hbar)$$

$$\frac{\partial\Gamma^{(2)}}{\partial g} = 0 + O(\hbar)$$

since $\beta_m, \gamma = O(\hbar)$, we find that at $O(\hbar)$ the RG equations give

$$0 = \left(\frac{\partial}{\partial \ln \mu} + \beta_m \frac{\partial}{\partial m^2} + \beta_g \frac{\partial}{\partial g} - 2\gamma_\phi \right) \tilde{\Gamma}^{(2)}(p, -p)$$

$$= -\hbar \frac{g^2}{(4\pi)^3} \left(m^2 + \frac{1}{6} p^2\right) - \beta_m$$

$$- 2\gamma_\phi \left(-p^2 - m^2\right)$$

$$= (p^2 + m^2)\left(2\gamma_\phi - \frac{1}{6}\hbar \frac{g^2}{(4\pi)^3}\right) - \frac{5}{6}\hbar \frac{g^2}{(4\pi)^3} m^2 - \beta_m$$

so that

$$\underline{\underline{\gamma_\phi = \frac{\hbar}{12} \frac{g^2}{(4\pi)^3}}} + O(\hbar^2)$$

$$\beta_m = \underline{\underline{-\frac{5}{6}\hbar \frac{g^2}{(4\pi)^3} m^2}} + O(\hbar)$$

To find β_g, we need to go to cubic order. There

$$\frac{\partial \tilde{\Gamma}^{(3)}}{\partial \ln \mu} = -\frac{\hbar}{2} \left(\frac{g}{4\pi}\right)^3 \cdot \int dF_3 \cdot 2 = -\hbar \left(\frac{g}{4\pi}\right)^3$$

$$\frac{\partial \tilde{\Gamma}^{(3)}}{\partial m^2} = 0 + O(\hbar)$$

$$\frac{\partial \tilde{\Gamma}^{(3)}}{\partial g} = -1 + O(\hbar)$$

Once again, $\beta_g = O(\hbar)$ we find that at $O(\hbar)$ the RG equations give

$$0 = \left(\frac{\partial}{\partial \ln \mu} + \beta_m \frac{\partial}{\partial m^2} + \beta_g \frac{\partial}{\partial g} - 3\gamma_\phi \right) \tilde{\Gamma}^{(3)}$$

$$= -\hbar \left(\frac{g}{4\pi}\right)^3 - \beta_g - 3\gamma_\phi \cdot - g$$

so

$$\beta_g = \hbar \left(\frac{g}{4\pi}\right)^3 \left(-1 + \frac{3}{12}\right) + O(\hbar)$$

$$= \underline{\underline{-\frac{3}{4}\hbar \left(\frac{g}{4\pi}\right)^3}} + O(\hbar^2)$$