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Special Topics: JT Gravity

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intro

We will describe how JT gravity and its universal Schwarzian dynamics arises. We also discuss boundary correlation functions for matter fields in the presence of these Schwarzian modes. We mostly follow [1–3] but also found [4] useful.

1 Dilaton and JT Gravity

Gravity in two dimensions does not propagate any dynamical bulk degrees of freedom — the equations of motion are trivially satisfied and the bulk action is a topological invariant. We can however find non-trivial dynamics if we allow for scalar-tensor theories with some dilaton degree of freedom Φ[1]. The most general dilaton gravity model we can write down with two-derivative action is of the form

$$S_{\rm DG}[g,\Phi] = \int_M d^2x\sqrt{|g|}\left[U_1(\tilde{\Phi})\frac{R}{2} + U_2(\tilde{\Phi})g^{\mu\nu}\partial_\mu\tilde{\Phi}\partial_\nu\tilde{\Phi} + U_3(\tilde{\Phi})\right] \tag{1.1}$$

for some arbitrary functions U_1, U_2, and U_3. However, we can

- Redefine $\Phi = U_1(\tilde{\Phi})$ provided that $U_1'(\tilde{\Phi}) \neq 0$,
- Weyl transform $g \to e^{2\omega}g$ with $\omega(x) = \int^{\Phi(x)} U_2(\Phi')d\Phi'$ to cancel the kinetic term,

to bring this into the form

$$S_{\rm DG}[g,\Phi] = \int_M d^2x\sqrt{|g|}\left[\Phi\frac{R}{2} - U(\Phi)\right]. \tag{1.2}$$

These types of theories are well-motivated, as they can arise e.g. in spherical/dimensional reductions from higher-dimensional gravity and critical strings.

[1] Also known to be generically equivalent to $f(R)$ theories.

JT gravity is a specific example of a dilaton gravity theory with $U(\Phi) = \Lambda\Phi$. Including the GHY boundary term and a holographic counterterm, the action is

$$S_{\rm JT}[g,\Phi] = \int_M d^2x\sqrt{-g}\Phi\left(\frac{R}{2} - \Lambda\right) + \int_{\partial M} dx\sqrt{h}\Phi(K-1) \tag{1.3}$$

where h is the induced metric and K is the extrinsic curvature on the boundary ∂M. This arises as the universal near-horizon limit of near-extremal black holes in higher dimensions, and is hence (more or less) of phenomenological relevance [2].

In first-order form, we can write the bulk action as

$$S_{\rm JT}[g,\Phi] = \int_M d^2x\,\Phi\left(d\omega[e] - \frac{\Lambda}{2}\epsilon^{ab}e_a \wedge e_b\right) + X^a\left(de_a + \epsilon_{ab}\omega[e]\wedge e_b\right). \tag{1.4}$$

The normalisations are chosen to be consistent with the previous action, and the auxiliary fields X^a ensure that the connection ω is torsion-free.

2 Classical Bulk Solutions

In this section, we will find the classical bulk solutions (essentially the saddle point) of the dilaton-gravity system, by solving the classical equations.

2.1 Gravity: Bulk AdS_2

The action is linear in the dilaton, so its equation of motion simply imposes

$$R = 2\Lambda = -2/L^2 \tag{2.1}$$

which we choose to negative, with L defining the AdS length scale. We will set $L = 1$ from now on. All two-dimensional manifolds are conformally flat, so this means our bulk spacetime is just AdS_2 [3]. We can write its metric in different coordinates as

$$\text{Poincaré: } ds^2 = \frac{-dt^2 + dz^2}{z^2} = -\frac{4dudv}{(u-v)^2}, \quad u = t - z, v = t + z \tag{2.2a}$$

$$\text{Global: } ds^2 = -\frac{4dUdV}{\sin(U-V)^2} = \frac{-dT^2 + dZ^2}{\sin^2(Z)}, \quad U = \frac{T-Z}{2} = \arctan u, V = \frac{T+Z}{2} = \arctan v \tag{2.2b}$$

$$\text{Rindler: } ds^2 = d\rho^2 - \sinh^2(\rho)d\tau^2, \quad t = \coth\rho\, e^{\tau}, z = \mathrm{csch}\rho\, e^{\tau} \tag{2.2c}$$

which cover different areas of the space. The isometry group of AdS_2 is $\mathrm{SO}(1,2) \simeq \mathrm{PSL}(2,\mathbb{R}) \simeq \mathrm{SL}(2,\mathbb{R})/\mathbb{Z}_2$, i.e. the Möbius transformations [4]. We will therefore want any observables to be invariant under this these transformations. One can verify that, indeed the metric in Poincaré coordinates is

[2] From the higher-dimensional perspective, the dilaton plays the role of an area, in which case the action should come with an appropriate factor of $M_{\rm Pl}^2$, which can however always be absorbed in Φ. We would then expect also a contribution from the background value of the dilaton Φ_0 multiplying the topological Einstein-Hilbert term. This usually contributes to the entropy.

[3] In two dimensions, the Riemann tensor is fully specified by the Ricci scalar and one can always find a conformal transformation to make this vanish.

[4] Note that $\mathrm{SO}(1,2)$ has trivial centre while $\mathrm{SL}(2,\mathbb{R})$ does not.

invariant under $(u,v) \mapsto (\frac{au+b}{cu+d}, \frac{av+b}{cv+d})$ for $ad-bc=1$. This means that an equivalent coordinate transformation to the Rindler patch is

$$t \pm z = \frac{a(\pm 1 + \cosh\rho)e^{\tau/2} + b\sinh\rho\, e^{-\tau/2}}{c(\pm 1 + \cosh\rho)e^{\tau/2} + d\sinh\rho\, e^{-\tau/2}}. \tag{2.3}$$

In particular, choosing $(a,b,c,d) = \frac{1}{\sqrt{2}}(1,-1,1,1)$, has the nice property that near the boundary ($z \to 0$ or $\rho \to \infty$), we get $t \sim \tanh(\tau/2)$. We will mostly be interested in Euclidean signature, which we reach by defining Euclidean time $t_E = it_L$. We get similar expressions for the metric in the different coordinates, which all now cover the entire hyperbolic disk $\mathbb{H}^2$.

When considering the field theory just on AdS_2 , we will find that the dilaton and (if present) bulk fields diverge near the boundary. This is unproblematic — in the context of holography these correspond to UV divergences in the CFT, which we know how to handle[5]. We should therefore study cutoff EAdS_2 as a cutout from the full space defined by a boundary parameterised by the Poincaré coordinates (t,z). Let u be a boundary time (no relation to the lightcone coordinate), so that $t = t(u)$ and $z = z(u)$. We fix the length of this boundary by demanding that its line element is given by

$$ds\big|_{\text{boundary}} = \frac{1}{\epsilon}du \tag{2.4}$$

for some UV cutoff $\epsilon \to 0^+$. We can write this is as

$$ds^2\big|_{\text{boundary}} = g_{\mu\nu}dx^\mu dx^\nu = g_{\mu\nu}\frac{dx^\mu}{du}\frac{dx^\nu}{du}du^2 \tag{2.5}$$

and recognise that it is equivalent to demanding that metric induced on the boundary (parameterised by just u) is fixed to be

$$h = g\big|_{\text{boundary}} = g_{\mu\nu}\frac{dx^\mu}{du}\frac{dx^\nu}{du} = \frac{t'(u)^2 + z'(u)^2}{z(u)^2} = \frac{1}{\epsilon^2}. \tag{2.6}$$

this uniquely fixes $z(u) = \epsilon t'(u) + \mathcal{O}(\epsilon^2)$ in terms of $t(u)$, so there is only a single dynamical boundary gravitational degree of freedom.

The original Einstein-Hilbert action enjoys full diffeomorphism invariance, and is in fact identical for all configurations. It includes arbitrary reparameterisations of the boundary time $u \mapsto f(u)$, that give rise to zero modes in the action. Allowing for cutouts means that the symmetry group is explicitly broken down to only a subgroup of these transformations — some will actually change the shape of the cutout while others, the asymptotic diffeomorphisms (e.g. rotations and translations) leave them invariant. These are precisely the $\mathrm{PSL}(2,\mathbb{R})$ transformations. In the limit where $\epsilon \to 0^+$, $z \to 0$, and we see that the dynamical mode $t(u)$ transforms as

$$t(u) \mapsto \frac{at(u)+b}{ct(u)+d}, \quad ad-bc = 1. \tag{2.7}$$

It is sometimes said that the $t(u)$-modes are the Goldstones modes for the corresponding breaking of the symmetry[6].

[5] In holographic renormalisation, we think of the radial direction of AdS as an energy scale in the CFT. For example, empty AdS corresponds to an RG flow trajectory that stays at the CFT. Turning on some irrelevant deformation in the CFT means that our trajectory gets deflected as we flow from towards the UV in the CFT, i.e. the boundary of AdS, in an uncontrolled manner. This is precisely what the non-normalisable modes correspond to on the AdS side. We know how to treat this with a UV cutoff and counterterms on the CFT side, which corresponds to an IR cutoff on the gravity side.

[6] The Mermin-Wagner theorem states that there are no Goldstone modes in one or two dimensions. This is not a spontaneous breaking of symmetry.

2.2 Dilaton

The equations of motion for the dilaton are obtained by varying the JT gravity action with respect to the metric. In the presence of matter with stress-tensor $T_{\mu\nu}$ this is just

$$T_{\mu\nu} = \nabla_\mu \nabla_\nu \Phi - g_{\mu\nu}(\Box + \Lambda)\Phi, \tag{2.8}$$

or in Euclidean Poincaré coordinates

$$T_{tt} = -\left(\frac{1}{z}\partial_z + \partial_z^2 + \frac{\Lambda}{z^2}\right)\Phi \tag{2.9a}$$

$$T_{tz} = \left(\frac{1}{z} + \partial_z\right)\partial_t\Phi \tag{2.9b}$$

$$T_{zz} = -\left(-\frac{1}{z}\partial_z + \partial_t^2 + \frac{\Lambda}{z^2}\right)\Phi. \tag{2.9c}$$

Let us solve this in the absence of matter, with $\Lambda = -1$. The (t, z)-component tells us that

$$\Phi(t, z) = \frac{f(t)}{z} + g(z) \tag{2.10}$$

for some arbitrary functions $f(t)$ and $g(z)$. Plugging this Ansatz into the other components gives us

$$\frac{1}{z^2}g(z) - \frac{1}{z}g'(z) - g''(z) = 0, \quad -\frac{1}{z^2}g(z) - \frac{1}{z}g'(z) + \frac{1}{z}f''(z) = 0 \tag{2.11}$$

which can be solved by

$$f(t) = \frac{b}{z^2}t^2 + bt + d, \quad g(z) = \frac{a}{z} + cz \tag{2.12}$$

for some arbitrary constants a, b, c, and d. Putting everything together, and relabelling $a + d \to a$, we find that

$$\Phi = \frac{a + bt + c(t^2 + z^2)}{z} = \frac{\tilde{a} + \tilde{b}(u + v) + \tilde{c}uv}{u - v} \tag{2.13}$$

where $(\tilde{a}, \tilde{b}, \tilde{c}) = (-2a, -b+2c)$. Further, note that the Möbius transformation $(u, v) \mapsto (\frac{Au+B}{Cu+D}, \frac{Av+V}{Cv+D})$ with $C = -\frac{1}{2B}$ and $D = \frac{1}{2A}$ (such that $AD - BC = 1$ and $AD + BC = 0$) sends

$$\frac{u + v}{u - v} \to \frac{1}{AB}\frac{B^2 - A^2uv}{u - v}, \tag{2.14}$$

effectively shifting $(\tilde{a}, \tilde{b}, \tilde{c}) \mapsto (\tilde{a} + B/A, 0, \tilde{c} - A/B)$. This means we can always effectively set $b = 0$.

This solution has some interesting properties. First, we note that it indeed diverges as we approach the boundary $z = 0$. For holographic renormalisation, we therefore set the boundary condition for the dilaton as

$$\Phi\big|_{\text{bdry}} = \lim_{z\to 0}\Phi \sim \frac{1}{\epsilon}\frac{a + ct^2(u)}{t'(u)} = \frac{\Phi_r}{\epsilon} \tag{2.15}$$

where Φ_r is the renormalised field. Furthermore, note that $\zeta^\mu = \epsilon^{\mu\nu}\partial_\nu\Phi$ satisfies

$$2\nabla_{(\mu}\zeta_{\nu)} = 2\epsilon_{\rho(\nu}\nabla^\rho\nabla_{\mu)}\Phi = 2\epsilon_{\rho(\nu}\delta^\rho_{\mu)}(\Box + \Lambda)\Phi = 2\epsilon_{(\mu\nu)}(\Box + \Lambda)\Phi = 0 \tag{2.16}$$

by the equations of motion. In other words ζ is a Killing of the dilaton-gravity system — this is not a boring observation because higher-genus surfaces do not admit Killing vectors.

3 Boundary Dynamics

In this section, we will derive the action that governs the dynamics of the boundary modes and then study them.

3.1 Boundary Schwarzian Action

We have seen that the classical bulk equations of motion fix the bulk metric to be AdS_2 while forcing the dilaton to blow up at the boundary. On this saddle point, we are left with the following Euclidean action

$$I_{\mathrm{JT}} = -\int_{\partial M} dx\sqrt{h}\,\Phi(K-1) = -\int_{\partial M} ds\,\Phi(K-1) = -\int_{\partial M} \frac{du}{\epsilon}\frac{\Phi_r}{\epsilon}(K-1). \tag{3.1}$$

Let us now compute this. First note that the unit normalised tangent and normal vector fields to the curve $(t(u), z(u))$ are given by

$$t^\mu = \frac{z}{\sqrt{t'^2+z'^2}}\begin{pmatrix} t'(u) \\ z'(u)\end{pmatrix}, \quad n^\mu = \frac{z}{\sqrt{t'^2+z'^2}}\begin{pmatrix} -z'(u) \\ t'(u)\end{pmatrix}. \tag{3.2}$$

Then, using the fact that the only non-zero Christoffel symbols of EAdS_2 in Poincaré coordinates are $-\Gamma^t{}_{zt} = \Gamma^z{}_{tt} = -\Gamma^z{}_{zz} = 1/z$, we find that the trace of the extrinsic curvature induced on the boundary curve is [7] [8]

$$\begin{aligned}
K =& h^{uu}K_{uu} = \left(g_{\mu\nu}\frac{dx^\mu}{du}\frac{dx^\nu}{du}\right)^{-1}\frac{dx^\alpha}{du}\frac{dx^\beta}{du}K_{\alpha\beta} = \left(g_{\mu\nu}\frac{dx^\mu}{du}\frac{dx^\nu}{du}\right)^{-1}\frac{dx^\alpha}{du}\frac{dx^\beta}{du}h^\rho_\alpha\nabla_\rho n_\beta \\
=& \left(g_{\mu\nu}\frac{dx^\mu}{du}\frac{dx^\nu}{du}\right)^{-1}\frac{dx^\alpha}{du}\frac{dx^\beta}{du}\left(\delta^\rho_\alpha - n_\alpha n^\rho\right)\nabla_\rho n_\beta \\
=& \left(g_{\mu\nu}\frac{dx^\mu}{du}\frac{dx^\nu}{du}\right)^{-1}\frac{dx^\alpha}{du}\frac{dx^\beta}{du}\left(\partial_\alpha n_\beta - \Gamma^\rho{}_{\alpha\beta}n_\rho\right) \\
=& \left(g_{\mu\nu}\frac{dx^\mu}{du}\frac{dx^\nu}{du}\right)^{-1}\left(\frac{dx^\beta}{du}\partial_u n_\beta - \Gamma^\rho{}_{\alpha\beta}\frac{dx^\alpha}{du}\frac{dx^\beta}{du}n_\rho\right) \\
=& \left(g_{\mu\nu}\frac{dx^\mu}{du}\frac{dx^\nu}{du}\right)^{-1}\left(t'\partial_u n_t + z'\partial_u n_z - \frac{1}{z}\left[-2t'z'n_t + t'^2 n_z - z'^2 n_z\right]\right) \\
=& \frac{t'\left(t'^2+z'^2+zz''\right) - zz't''}{\left(t'^2+u'^2\right)^{3/2}}.
\end{aligned} \tag{3.3}$$

Near the AdS_2 boundary, $z = \epsilon t' + \mathcal{O}(\epsilon^2)$,

$$K = 1 + \epsilon^2 (St)(u) + \mathcal{O}(\epsilon^4) \tag{3.4}$$

where the Schwarzian derivative is given by

$$(Sf)(z) = -\frac{1}{2}\left(\frac{f''}{f'}\right)^2 + \left(\frac{f''}{f'}\right)' = \frac{-3f''^2 + 2f'f'''}{2f'^2}. \tag{3.5}$$

This means that dynamics of the boundary degrees of freedom $t(u)$ is governed by the boundary Schwarzian action

$$I_{\mathrm{Schw}} = -\int_{\partial M} du\,\Phi_r(u)(St)(u). \tag{3.6}$$

[7]The surface is one-dimensional, so the tensor only has one component.
[8]This calculation looks deceptively simple!

The bulk and boundary equations of motion for the dilaton are equivalent to each other, so the solution for the renormalised dilaton field is simply the boundary limit of the bulk dilaton solution. For convenience, we redefine our notion of boundary time with $d\tilde{u} = \frac{\bar{\Phi}}{\Phi_r} du$, under which,

$$(St)(u) = \tilde{u}'^2 (St)(\tilde{u}) + (S\tilde{u})(u) \tag{3.7}$$

so that

$$\begin{aligned} I_{\text{Schw}} &= -\int_{\partial M} d\tilde{u} u' \, \Phi_r(u)(St)(u) \\ &= -\int_{\partial M} d\tilde{u} \left[\tilde{u}' \Phi_r(u)(St)(\tilde{u}) + \tilde{u}'^{-1} \, \Phi_r(u)(S\tilde{u})(u)\right] \\ &= -\int_{\partial M} d\tilde{u} \left[\tilde{u}' \Phi_r(u)(St)(\tilde{u}) + \tilde{u}'^{-1} \, \Phi_r(u) \frac{\Phi_r'^2 - 2\Phi_r''\Phi_r}{2\Phi_r^{-2}}\right] \\ &= -\int_{\partial M} d\tilde{u} \left[\bar{\Phi}(St)(\tilde{u}) + \frac{1}{2\bar{\Phi}} \left(\Phi_r'^2 - 2\Phi_r''\Phi_r\right)\right]. \end{aligned} \tag{3.8}$$

We need to fix a boundary condition for the dilaton Φ_r to be constant [9], in which case the above action reduces to

$$I_{\text{Schw}} = -\bar{\Phi} \int_{\partial M} d\tilde{u} (St)(\tilde{u}). \tag{3.9}$$

Fixing the asymptotics of the dilaton like this leaves freedom only in the boundary time parameter, so we will now turn to study the dynamics of these.

3.2 The Schwarzian Derivative

From the definition given above, we can immediately verify that the chain rule for the Schwarzian derivative is

$$(S[f \circ g])(z) = g'^2 (Sf)[g(z)] + (Sg)(z). \tag{3.10}$$

We can also use the following function of two variables

$$F(z, w) = \log\left(\frac{f(z) - f(w)}{z - w}\right) \tag{3.11}$$

as an alternative way to define the Schwarzian derivative, as

$$(Sf)(w) = 6 \frac{\partial^2 F(z, w)}{\partial z \partial w}\bigg|_{z=w} = 6 \lim_{z \to w} \left(\frac{f'(z) f'(w)}{(f(z) - f(w))^2} - \frac{1}{(z - w)^2}\right). \tag{3.12}$$

Using this, it is easy to see that for $M(z) = \frac{az+b}{cz+d}$,

$$\begin{aligned} (SM)(w) =& 6 \lim_{z \to w} \left(\frac{\frac{ad-bc}{(cz+d)^2} \frac{ad-bc}{(cw+d)^2}}{\left(\frac{az+b}{cz+d} - \frac{aw+b}{cw+d}\right)^2} - \frac{1}{(z - w)^2}\right) = 6 \lim_{z \to w} \left(\frac{(ad - bc)^2}{\left[(az + b)(cw + d) - (aw + b)(cz + d)\right]^2} - \frac{1}{(z - w)^2}\right) \\ =& 6 \lim_{z \to w} \left(\frac{(ad - bc)^2}{\left[(ad - bc)(z - w)\right]^2} - \frac{1}{(z - w)^2}\right) = 0 \end{aligned} \tag{3.13}$$

[9] From a higher-dimensional perspective, the dilaton value at the boundary corresponds to the radial the area of fixed radial hypersurfaces, so it is natural to fix this.

so the Schwarzian derivatives vanishes for $\mathrm{PSL}(2,\mathbb{R})$ transformations. Now, $(Sf)(z)=0$ is a third-order differential equation, so three initial conditions are required to find the unique solution. These map precisely into the four parameters (subject to one constraint) of $\mathrm{PSL}(2,\mathbb{R})$ via

$$(M(0), M'(0), M''(0)) = \left(\frac{b}{d}, \frac{1}{d^2}, -2\frac{c}{d^3}\right), \tag{3.14}$$

so that $f=M$ is the unique solution to $(Sf)(z)=0$.

Using the composition law for the Schwarzian derivative, we then find that for two functions $f(z)$ and $g(z)$ related by a Möbius transformation $f = M\circ g$,

$$(Sf)(z) = (S[M\circ g])(z) = g'^2(SM)(z) + (Sg)(z) = (Sg)(z), \tag{3.15}$$

so their Schwarzian derivatives are equal. Conversely, two functions with equal Schwarzian derivative must be related by a Möbius transformation. In other words, it is $\mathrm{PSL}(2,\mathbb{R})$ invariant!

3.3 Saddle Point and Fluctuations

We can minimally couple the Schwarzian action to matter in the following hybrid way

$$I = -\bar{\Phi}_r \int du\, (St)(u) + \int dtdz\, \mathcal{L}_m(\phi, \partial_t\phi) = -\bar{\Phi}_r \int du\, (St)(u) + \int dudz\, t'\mathcal{L}_m(\phi, \partial_t\phi). \tag{3.16}$$

The pure matter variation is

$$\delta I \subset \int dtdz \left(\frac{\partial \mathcal{L}_m}{\partial\phi}\delta\phi + \frac{\partial \mathcal{L}_m}{\partial\partial_t\phi}\delta\partial_t\phi\right) = \int dtdz \left(\frac{\partial \mathcal{L}_m}{\partial\phi} - \partial_t\frac{\partial \mathcal{L}_m}{\partial\partial_t\phi}\right)\delta\phi \tag{3.17}$$

which just gives the standard Euler-Lagrange variation

$$\frac{\partial \mathcal{L}_m}{\partial\phi} - \partial_t\frac{\partial \mathcal{L}_m}{\partial\partial_t\phi} = 0. \tag{3.18}$$

However, varying with respect to gravity in terms of the boundary parameter u gives the following [10]

$$\begin{aligned}
\delta I &= \int du \left(-\bar{\Phi}\partial_t(St)(u)\delta t + \int dz\, \left[\delta t'\mathcal{L}_m + t'\frac{\partial\mathcal{L}_m}{\partial t}\delta t\right]\right)\\
&= \int du \left(-\bar{\Phi}\frac{1}{t'}\partial_u(St)(u)\delta t + \int dz\, \left[\delta t'\mathcal{L}_m + t'\left(\frac{\partial\mathcal{L}_m}{\partial\phi}\partial_t\phi + \frac{\partial\mathcal{L}_m}{\partial\partial_t\phi}\partial_t^2\phi\right)\delta t\right]\right)\\
&= \int du \left(-\bar{\Phi}\frac{[(St)(u)]'}{t'}\delta t + \int dz\, \left[\delta t'\mathcal{L}_m + t'\partial_t\left(\frac{\partial\mathcal{L}_m}{\partial\partial_t\phi}\partial_t\phi\right)\delta t\right]\right)\\
&= \int du \left(-\bar{\Phi}\frac{[(St)(u)]'}{t'}\delta t + \int dz\, \delta t'\left[\mathcal{L}_m - \frac{\partial\mathcal{L}_m}{\partial\partial_t\phi}\partial_t\phi\right]\right)\\
&= \int du\, \delta t \left(-\bar{\Phi}\frac{[(St)(u)]'}{t'} - \partial_u\int dz\, \left[\mathcal{L}_m - \frac{\partial\mathcal{L}_m}{\partial\partial_t\phi}\partial_t\phi\right]\right).
\end{aligned} \tag{3.19}$$

This leads to the classical equations of motion

$$\bar{\Phi}\frac{[(St)(u)]'}{t'(u)} = -\partial_u\int dz\left(\mathcal{L}_m - \frac{\partial\mathcal{L}_m}{\partial\partial_t\phi}\partial_t\phi\right) = t'\frac{dH}{dt} \tag{3.20}$$

[10] This is, once again, deceptively simple-looking!

for the boundary Schwarzian mode, where $H = \int dz\, T_{tt}$ is the Hamiltonian of the matter sector.

We can solve this in the absence of matter, when $H = 0$. The solutions to this are functions with constant Schwarzian derivative, but we do not want to consider the trivial cases of the constant function and $\mathrm{PSL}(2,\mathbb{R})$ (gauge) transformations. To find the non-trivial solutions, we switch to Euclidean Rindler time near the boundary $t(u) = i \tanh \frac{-i\tau(u)}{2} = \tan \frac{\tau(u)}{2}$, with which

$$(S[t \circ \tau])(u) = \tau'^2 (St)(\tau) + (S\tau)(u) = (S\tau)(u) + \frac{1}{2}\tau'^2. \tag{3.21}$$

We therefore see that Rindler time τ linear in boundary time is a non-trivial solution to the equations of motion. Since we identify $\tau \sim \tau + 2\pi$ in Euclidean signature, we take this solution to be

$$\tau(u) = \frac{2\pi}{\beta} u \tag{3.22}$$

so that $u \sim u + \beta$ is also identified. However, for simplicity we will set $\beta = 2\pi$ from now on. Note that this is however a fourth-order differential equation, and the solutions are not generally known.

We can also go beyond the semiclassical regime and study gravitational dynamics by considering fluctuations around the saddle point

$$\tau(u) = u + \varepsilon(u). \tag{3.23}$$

Tree-level gravitational effects are described by linearised fluctuations, so we want to expand the action to quadratic order. Once again, switching to Rindler time

$$(St)(u) = (S\tau)(u) + \frac{1}{2}\tau'(u)^2 = \frac{1}{2} + \epsilon\left(\varepsilon' + \varepsilon''\right) + \frac{1}{2}\varepsilon^2\left(\varepsilon^2 - 3\varepsilon''^2 - 2\varepsilon'\varepsilon'''\right) + \mathcal{O}\left(\varepsilon^3\right) \tag{3.24}$$

and dropping constant and/or total derivative terms

$$I^{(2)}_{\text{Schw}} = -\frac{1}{2}\bar{\Phi}\int_{\partial M} du\, (\varepsilon'^2 - \varepsilon''^2). \tag{3.25}$$

The boundary ∂M is parameterised by $u \sim u + 2\pi \in [0, 2\pi)$, so we can expand in terms of Fourier modes

$$\varepsilon(u) = \sum_{n\in\mathbb{Z}} e^{-inu}\varepsilon_n, \quad \varepsilon_n = \frac{1}{2\pi}\int_0^{2\pi} du\, e^{inu}\varepsilon(u) \tag{3.26}$$

which diagonalises the quadratic action

$$I^{(2)}_{\text{Schw}} = -\frac{1}{2}\bar{\Phi}\sum_{n\in\mathbb{Z}}(n^2 - n^4)\varepsilon_n\varepsilon_{-n} = -\frac{1}{2}\bar{\Phi}\sum_{n\in\mathbb{Z}} n^2(1-n^2)\varepsilon_n\varepsilon_{-n}. \tag{3.27}$$

From the definition of the 1PI effective action in Lorentzian signature, we find that in Euclidean signature $[G^{(2)}]^{-1} = i\Gamma^{(2)} \to -\Gamma^{(2)}$, so we should be able to read off the two-point function from $S \sim \Gamma$ at tree-level. However, we need to mod out the $\mathrm{PSL}(2,\mathbb{R})$ gauge redundancies in the path integral. These give rise to the zero modes in the action, with $n = -1, 0, +1$ (or ε_{-1}, ε_0, and ε_1) corresponding precisely to the three parameters/generators of the $\mathrm{PSL}(2,\mathbb{R})$ group[11]. We therefore end up with the following expression

$$\langle\varepsilon(u)\varepsilon(0)\rangle = \sum_{n\neq 0,\pm 1} e^{-inu}\langle\varepsilon_n\varepsilon_{-n}\rangle = \frac{1}{\bar{\Phi}}\sum_{n\neq 0,\pm 1}\frac{e^{-inu}}{n^2(1-n^2)} \tag{3.28}$$

[11] Show this explicitly.

for the coordinate-space two-point function. To evaluate this, we note that

$$f(s) = \frac{1}{e^{-2\pi i s} - 1} \frac{e^{-isu}}{s^2(1-s^2)} \tag{3.29}$$

has poles at $s \in \mathbb{Z}$ with the desired residue. In particular

$$\oint_{\mathcal{C}} ds\, f(s) = \sum_{n\neq 0,\pm 1} \frac{e^{-inu}}{n^2(1-n^2)} \tag{3.30}$$

for some contour $\mathcal{C}$ picking up only poles at $n \neq 0, \pm 1 \in \mathbb{Z}$. We can take $\mathcal{C}$ to be the union of a counter-clockwise circular contour $\mathcal{C}_1$ encompassing all $s \in \mathbb{Z}$ and another clockwise contour $\mathcal{C}_2$ only encircling $s = -1, 0, +1$. It turns out that the integral over $\mathcal{C}_2$ vanishes. To see this, we break $\mathcal{C}_2$ into two pieces $\mathcal{C}_2^{\pm}$ in the upper and lower half planes. For large $|s|$ with $s = x + iy$,

$$f(s) \sim \frac{e^{-is(u-2\pi)}}{s^2(1-s^2)} \sim \frac{e^{-ix(u-2\pi)+y(u-2\pi)}}{s^2(1-s^2)} \tag{3.31}$$

and since the original expression is symmetric under $u \to -u$, we can pick $u > 0 \sim 2\pi$ or $u < 0 \sim 2\pi$ for $\mathcal{C}_2^-$ or $\mathcal{C}_2^+$ respectively. At large $|s|$, the integrals over both vanish, and we are just left with a contour integral over $\mathcal{C}_1$ which picks up (minus) the residues at $s = -1, 0, +1$ so that

$$\langle \varepsilon(u)\varepsilon(0)\rangle = \frac{1}{\bar{\Phi}}\left[(u-\pi)\sin(u) + \frac{5}{2}\cos(u) - \frac{(u-\pi)^2}{2} + 1 + \frac{\pi^2}{6}\right], \quad u > 0. \tag{3.32}$$

By symmetry, this expression holds for $u < 0$ when we just replace $u \mapsto |u|$, so that generally

$$\langle \varepsilon(u_i)\varepsilon(u_j)\rangle = \frac{1}{\bar{\Phi}}\left[(|u_{ij}|-\pi)\sin(|u_{ij}|) + \frac{5}{2}\cos(|u_{ij}|) - \frac{(|u_{ij}|-\pi)^2}{2} + 1 + \frac{\pi^2}{6}\right] \tag{3.33}$$

where $u_{ij} = u_i - u_j$. It is also useful to note that

$$\langle \varepsilon'(u_i)\varepsilon(u_j)\rangle = \mathrm{sgn}(u_{ij})\frac{\partial}{\partial |u_{ij}|}\langle \varepsilon(u_i)\varepsilon(u_j)\rangle \tag{3.34}$$

$$\langle \varepsilon'(u_i)\varepsilon'(u_j)\rangle = -\frac{\partial^2}{\partial |u_{ij}|^2}\langle \varepsilon(u_i)\varepsilon(u_j)\rangle\,. \tag{3.35}$$

4 Matter Coupling

Let us couple the theory to some matter and study its boundary dynamics with gravity. This can be done using the usual holographic dictionary, where the on-shell bulk partition function is identified with the boundary field theory generating functional [5]

$$Z_{\text{Bulk/Gravity}}[\phi] = Z_{\text{Boundary/QFT}}[\phi] \tag{4.1}$$

or

$$\int_{\Phi|_{\partial M}\sim\phi} \mathcal{D}\Phi\, e^{-I[\Phi]} = \left\langle e^{-\int_{\partial M}\mathcal{O}\phi}\right\rangle_{\text{QFT}}. \tag{4.2}$$

In the saddle point approximation this is the statement that the on-shell gravitational action supplied with appropriate boundary conditions is equal to the Wilsonian effective action sourced by the boundary fields.

Let us consider the simplest case of a massive scalar field with Euclidean action

$$I_m = -\frac{1}{2}\int d^2x\sqrt{g}\left(g^{\mu\nu}\partial_\mu\chi\partial_\nu\chi + m^2\chi^2\right). \tag{4.3}$$

To find boundary correlation functions, we only need to take functional derivatives of the boundary partition function $Z[\chi_r]$ with respect to χ_r. On the bulk side, we integrate out the dilaton and then matter (in this order!) to find

$$Z[\chi_r] \sim \int_{\Phi_r,\chi_r} \mathcal{D}g\,\mathcal{D}\Phi\,\mathcal{D}\chi\, e^{-I_{\rm JT}[g,\Phi]-I_m[\chi,g]} \sim \int \mathcal{D}t\, e^{-I_{\rm Schw}[t]} Z_m[\chi_r,t]. \tag{4.4}$$

In the saddle point approximation on the gravitational side, this reduces to

$$Z_{\rm Saddle}[\chi_r] \sim e^{-I^{\text{on-shell}}_{\rm Schw}[t]} Z_m[\chi_r,t] \tag{4.5}$$

evaluated at the combined saddle. Note that, even at tree-level, the saddle of $t(u)$ will also depend on the matter sector and is therefore non-trivial. However, in a regime where the gravitational dynamics are dominant, we can ignore the backreaction from the matter sector and determine the saddle for $t(u)$ independently[12] Drop this assumption?.

4.1 Pure Matter Sector

We can find $Z_m[\chi_r,t]$ exactly. Solving the bulk equations of motion asymptotically near the boundary, we find

$$\chi(t,z) = z^{1-\Delta}\tilde{\chi}_r(t) + \dots \tag{4.6}$$

where ellipses denote asymptotically subleading terms as $z \to 0$, and

$$\Delta = \frac{1}{2} + \sqrt{\frac{1}{4}+m^2} = \frac{1}{2}\left(1+\sqrt{1+4m^2}\right). \tag{4.7}$$

In our putative dual theory we think of χ_r as sources for operators with scaling dimension Δ. The holographic dictionary (at tree-level) tells us that the boundary field theory partition function will be determined by the on-shell bulk action. In particular, the quadratic part is[13]

$$\log Z_m[\chi_r,t] \supset D\int dt\,dt'\,\frac{\tilde{\chi}_r(t)\tilde{\chi}_r(t')}{|t-t'|^{2\Delta}} \tag{4.8}$$

where

$$D = \frac{(\Delta-\frac{1}{2})\Gamma(\Delta)}{\sqrt{\pi}\Gamma(\Delta-\frac{1}{2})} = \frac{1}{2}c_\Delta\big|_{d=1} \tag{4.9}$$

and

$$c_\Delta = (2\Delta-d)\frac{\Gamma(\Delta)}{\pi^{d/2}\Gamma\left(\Delta-\frac{d}{2}\right)} \tag{4.10}$$

is the correct normalisation for scalar two-point functions when $\Delta = d/2+k$ and $k \in \mathbb{Z}$[5].

[12] From a higher-dimensional perspective, the Schwarzian action comes with extra factors of $M^2_{\rm Pl}$ so for sufficiently slowly growing scaling dimensions, this is satisfied.

[13] This is slightly subtle. The boundary is not located at fixed $z=\epsilon$, so we need to evaluate the boundary term with respect to the vector n^μ that is normal to the wiggling boundary curve $(t(u), z(u))$. The corrections are subleading in the UV cut-off ϵ.

When we consider the free matter theory, this is in fact the only term in the effective action $W_m = -\log Z_m$, but there will be other contributions to the connected correlation functions when we account for gravitational fluctuations. As discussed previously, these are contained in the AdS_2 boundary and are therefore hidden in the (t, z)-dependence. Defining $\chi_r(u) = t'(u)^{1-\Delta}\tilde{\chi}_r(t)$ such that the asymptotic boundary condition becomes

$$\chi(t, z) = z^{1-\Delta}\tilde{\chi}_r(t) + \cdots = [\epsilon t'(u)]^{1-\Delta}\tilde{\chi}_r(t) + \cdots = \epsilon^{1-\Delta}\chi_r(u) + \ldots, \tag{4.11}$$

we see that

$$\log Z_m[\chi_r, t] \supset D \int du\, du' \left[\frac{t'(u)t'(u')}{(t(u) - t(u'))^2}\right]^{\Delta} \chi_r(u)\chi_r(u') \tag{4.12}$$

with everything evaluated on the saddle.

4.2 Boundary Correlation Functions

In the regime where the gravitational dynamics are dominant, there are no matter self-interactions via gravity and the operator $\mathcal{O}_\chi$ dual to χ_r is essentially free, and the only non-vanishing correlation function is the 2-point function at tree-level

$$\langle \mathcal{O}_\chi(u_1)\mathcal{O}_\chi(u_2)\rangle \sim \left[\frac{t'(u_1)t'(u_2)}{(t(u_1) - t(u_2))^2}\right]^{\Delta} \tag{4.13}$$

with $t(u)$ on the saddle of the purely Schwarzian theory.

We can capture gravitational loops by expanding around the Schwarzian saddle point

$$t(u) = \tan\frac{u + \varepsilon(u)}{2} \tag{4.14}$$

which gives

$$\left[\frac{t'(u_1)t'(u_2)}{(t(u_1) - t(u_2))^2}\right]^{\Delta} = \frac{1}{\left(2\sin\frac{u_{12}}{2}\right)^{2\Delta}}\left[1 + \mathcal{B}(u_1, u_2) + \mathcal{C}(u_1, u_2) + \mathcal{O}(\varepsilon^3)\right] \tag{4.15}$$

where $u_{ij} = u_i - u_j$ and the higher-loop corrections are captured in

$$\mathcal{B}(u_1, u_2) = \Delta\left(\varepsilon'(u_1) + \varepsilon'(u_2) - \frac{\varepsilon(u_1) - \varepsilon(u_2)}{\tan\frac{u_{12}}{2}}\right) \tag{4.16}$$

$$\begin{aligned}\mathcal{C}(u_1, u_2) = \frac{\Delta}{(2\sin\frac{u_{12}}{2})^2}\Big[&(1 + \Delta + \Delta\cos u_{12})\big[\varepsilon(u_1) - \varepsilon(u_2)\big]^2 \\ &- 2\Delta\sin u_{12}\big[\varepsilon(u_1) - \varepsilon(u_2)\big]\big[\varepsilon'(u_1) + \varepsilon'(u_2)\big] \\ &- (1 - \cos u_{12})\Big((1 - \Delta)\big[(\varepsilon'(u_1)^2 + \varepsilon'(u_2)^2)\big] - 2\Delta\varepsilon'(u_1)\varepsilon'(u_2)\Big)\Big].\end{aligned} \tag{4.17}$$

We can then simply use our previous result for $\langle\epsilon(u_1)\epsilon(u_2)\rangle$ for Wick contractions to evaluate corrections due to the Schwarzian. In particular, expanding up to quadratic order in fluctuations, we find that the Wilson effective action takes the form

$$\begin{aligned}\log Z[\chi_r] \sim \int \mathcal{D}t\, e^{-I_{\rm Schw}[t]}\Bigg[D\left(\prod_{i=1}^{2}\int du_i\,\chi_r(u_i)\right)\frac{1}{\left(2\sin\frac{u_{12}}{2}\right)^{2\Delta}}\left[1+\mathcal{C}(u_1,u_2)\right] \\ +\frac{D^2}{2}\left(\prod_{i=1}^{4}\int du_i\,\chi_r(u_i)\right)\frac{1}{\left(2\sin\frac{u_{12}}{2}\right)^{2\Delta}\left(2\sin\frac{u_{34}}{2}\right)^{2\Delta}}\mathcal{B}(u_1,u_2)\mathcal{B}(u_3,u_4) \\ +\dots\Bigg].\end{aligned} \tag{4.18}$$

This means that

$$\langle \mathcal{O}_\chi(u_1)\mathcal{O}_\chi(u_2)\rangle \sim \frac{1}{\left(2\sin\frac{u_{12}}{2}\right)^{2\Delta}}\left[1+\langle\mathcal{C}(u_1,u_2)\rangle\right] \tag{4.19}$$

where

and

$$\langle\prod_{i=1}^{4}\mathcal{O}_\chi(u_i)\rangle \sim \frac{1}{\left(2\sin\frac{u_{12}}{2}\right)^{2\Delta}\left(2\sin\frac{u_{34}}{2}\right)^{2\Delta}}\langle\mathcal{B}(u_1,u_2)\mathcal{B}(u_3,u_4)\rangle \tag{4.20}$$

which can be evaluated explicitly. do this

References

[1] G. Sárosi, *AdS$_2$ holography and the SYK model*, *PoS* **Modave2017** (2018) 001 [1711.08482].

[2] D.A. Trunin, *Pedagogical introduction to the Sachdev–Ye–Kitaev model and two-dimensional dilaton gravity*, *Usp. Fiz. Nauk* **191** (2021) 225 [2002.12187].

[3] T.G. Mertens and G.J. Turiaci, *Solvable Models of Quantum Black Holes: A Review on Jackiw-Teitelboim Gravity*, 2210.10846.

[4] D. Grumiller, W. Kummer and D.V. Vassilevich, *Dilaton gravity in two-dimensions*, *Phys. Rept.* **369** (2002) 327 [hep-th/0204253].

[5] K. Skenderis, *Lecture notes on holographic renormalization*, *Class. Quant. Grav.* **19** (2002) 5849 [hep-th/0209067].